Differential forms on singular spaces, the minimal model program, and hyperbolicity of moduli stacks

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In memory of Eckart Viehweg

Abstract. The Shafarevich Hyperbolicity Conjecture, proven by Arakelov and Parshin, considers a smooth, projective family of algebraic curves over a smooth quasi-projective base curve Y. It asserts that if Y is of special type, then the family is necessarily isotrivial.

This survey discusses hyperbolicity properties of moduli stacks and generalisations of the Shafarevich Hyperbolicity Conjecture to higher dimensions. It concentrates on methods and results that relate moduli theory with recent progress in higher dimensional birational geometry.

Contents

Received by the editors December 21, 2011.

2000 *Mathematics Subject Classification.* Primary 14D22; Secondary 14D05.

Key words and phrases. hyperbolicity properties of moduli spaces, minimal model program, differential forms on singular spaces.

1. Introduction

1.1. The Shafarevich hyperbolicity conjecture

1.1.1. Statement In his contribution to the 1962 International Congress of Mathematicians, Igor Shafarevich formulated an influential conjecture, considering smooth, projective families $f^\circ : X^\circ \to Y^\circ$ of curves of genus $g > 1$, over a fixed smooth quasi-projective base curve Y°. One part of the conjecture, known as the "hyperbolicity conjecture", gives a sufficient criterion to guarantee that any such family is isotrivial. The conjecture was shown in two seminal works by Parshin and Arakelov, including the following special case.

Theorem 1.1 (Shavarevich Hyperbolicity Conjecture, [Sha63], [Par68, Ara71])**.** *Let $f^\circ : X^\circ \to Y^\circ$ be a smooth, complex, projective family of curves of genus $g > 1$, over a smooth quasi-projective base curve Y°. If Y° is isomorphic to one of the following varieties,*

- *the projective line $\mathbb{P}^1$,*
- *the affine line $\mathbb{A}^1$,*
- *the affine line minus one point $\mathbb{C}^*$, or*
- *an elliptic curve,*

then any two fibres of f° are necessarily isomorphic.

Notation-Assumption 1.2. *Throughout this paper, a* family *is a flat morphism of algebraic varieties with connected fibres. We always work over the complex number field.*

Remark 1.3. Following standard convention, we refer to Theorem 1.1 as "Shafarevich hyperbolicity conjecture" rather than "Arakelov-Parshin theorem". The reader interested in a complete picture is referred to [Vie01, p. 253ff], where all parts of the Shafarevich conjecture are discussed in more detail.

Formulated in modern terms, Theorem 1.1 asserts that any morphism from a smooth, quasi-projective curve Y° to the moduli stack of algebraic curves is necessarily constant if Y° is one of the special curves mentioned in the theorem. If Y° is a quasi-projective variety of arbitrary dimension, then any morphism from Y° to the moduli stack contracts all rational and elliptic curves, as well as all affine lines and $\mathbb{C}^*$s that are contained in Y°.

We refer to [HK10, Sect. 16.E.1] for a discussion of the relation between the Shafarevich hyperbolicity conjecture and the notions of Brody– and Kobayashi hyperbolicity.

1.1.2. Aim and scope This survey is concerned with generalisations of the Shafarevich hyperbolicity conjecture to higher dimensions, concentrating on methods and results that relate moduli– and minimal model theory. We hope that the methods presented here will be applicable to a much wider ranges of problems, in moduli theory and elsewhere. The list of problems that we would like to address include the following.

Questions 1.4. Apart from the quasi-projective curves mentioned above, what other varieties admit only constant maps to the moduli stack of curves? What about moduli stacks of higher dimensional varieties? Given a variety Y°, is there a good geometric description of the subvarieties that will always be contracted by any morphism to any reasonable moduli stack?

Much progress has been achieved in the last years and several of the questions can be answered today. It turns out that there is a close connection between the minimal model program of a given quasi-projective variety Y°, and its possible morphisms to moduli stacks. Some of the answers obtained are in fact consequences of this connection.

In the limited number of pages available, we say almost nothing about the history of higher dimensional moduli, or about the large body of important works that approach the problem from other points of view. Hardly any serious attempt is made to give a comprehensive list of references, and the author apologises to all those whose works are not adequately represented here, be it because of the author's ignorance or simply because of lack of space.

The reader who is interested in a broader overview of higher dimensional moduli, its history, complete references, and perhaps also in rigidity questions for morphisms to moduli stacks is strongly encouraged to consult the excellent surveys found in this handbook and elsewhere, including [HK10, Kov06, Vie01]. A gentle and very readable introduction to moduli of higher dimensional varieties in also found in [Kov09], while Viehweg's book [Vie95] serves as a standard technical reference for the construction of moduli spaces.

Most relevant notions and facts from minimal model theory can either be found in the introductory text [Mat02], or in the extremely clear and well-written reference book [KM98]. Recent progress in minimal model theory is surveyed in [HK10].

1.2. Outline of this paper

Section 2 introduces a number of conjectural generalisations of the Shafarevich hyperbolicity conjecture and gives an overview of the results that have been

obtained in this direction. In particular, we mention results relating the moduli map and the minimal model program of the base of a family.

Sections 3 and 4 introduce the reader to methods that have been developed to attack the conjectures mentioned in Section 2. While Section 3 concentrates on positivity results on moduli spaces and on Viehweg and Zuo's construction of (pluri-)differential forms on base manifolds of families, Section 4 summarises results concerning differential forms on singular spaces. Both sections contain sketches of proofs which aim to give an idea of the methods that go into the proofs, and which might serve as a guideline to the original literature. The introduction to Section 4 motivates the results on differential forms by explaining a first strategy of proof for a special case of a (conjectural) generalisation of the Shafarevich hyperbolicity conjecture. Following this plan of attack, a more general case is treated in the concluding Section 5, illustrating the use of the methods introduced before.

1.3. Acknowledgements

This paper is dedicated to the memory of Eckart Viehweg. Like so many other mathematicians of his age group, the author benefited immensely from Eckart's presence in the field, his enthusiasm, guidance and support. Eckart will be remembered as an outstanding mathematician, and as a fine human being.

The work on this paper was partially supported by the DFG Forschergruppe 790 "Classification of algebraic surfaces and compact complex manifolds". Patrick Graf kindly read earlier versions of this paper and helped to remove several errors and misprints. Many of the results presented here have been obtained in joint work of Sándor Kovács and the author. The author would like to thank Sándor for innumerable discussions, and for a long lasting collaboration. He would also like to thank the anonymous referee for careful reading and for numerous suggestions that helped to improve the quality of this survey.

Not all the material presented here is new, and some parts of this survey have appeared in similar form elsewhere. The author would like to thank his coauthors for allowing him to use material from their joint research papers. The first subsection of every chapter lists the sources that the author was aware of.

2. Generalisations of the Shafarevich hyperbolicity conjecture

2.1. Families of higher dimensional varieties

Given its importance in algebraic and arithmetic geometry, much work has been invested to generalise the Shafarevich hyperbolicity conjecture, Theorem 1.1. Historically, the first generalisations have been concerned with families $f^\circ : X^\circ \to Y^\circ$ where Y° is still a quasi-projective curve, but where the fibres of f° are allowed to have higher dimension. The following elementary example shows, however, that

Theorem 1.1 cannot be generalised naïvely, and that additional conditions must be posed.

Example 2.1 (Counterexample to the Shafarevich hyperbolicity conjecture for higher dimensional fibers)*.* Consider a smooth projective surface Y of general type which contains a rational or elliptic curve $C \subset Y$. Assume that the automorphism group of Y fixes the curve C pointwise. Examples can be obtained by choosing any surface of general type and then blowing up sufficiently many points in sufficiently general position —each blow-up will create a rational curve and lower the number of automorphisms. Thus, if c_1 and $c_2 \in C$ are any two distinct points, then the surfaces Y_{c_i} obtained by blowing up the points c_i are non-isomorphic.

In order to construct a proper family, consider the product $Y \times C$ with its projection $\pi : Y \times C \to C$ and with the natural section $\Delta \subset Y \times C$. If X is the blow-up of $Y \times C$ in Δ, then we obtain a smooth, projective family $f : X \to C$ of surfaces of general type, with the property that no two fibres are isomorphic.

It can well be argued that Counterexample 2.1 is not very natural, and that the fibres of the family f would trivially be isomorphic if they had not been blown up artificially. This might suggest to consider only families that are "not the blow-up of something else". One way to make this condition is precise is to consider only *families of minimal surfaces*, i.e., surfaces F whose canonical bundle K_F is semiample. In higher dimensions, it is often advantageous to impose a stronger condition and consider only *families of canonically polarised manifolds*, i.e., manifolds F whose canonical bundle K_F is ample.

Hyperbolicity properties of families of minimal surfaces and families of minimal varieties have been studied by a large number of people, including Migliorini [Mig95], Kovács [Kov96, Kov97] and Oguiso-Viehweg [OV01]. For families of canonically polarised manifolds, the analogue of Theorem 1.1 has been shown by Kovács in the algebraic setup [Kov00]. Combining algebraic arguments with deep analytic methods, Viehweg and Zuo prove a more general Brody hyperbolicity theorem for moduli spaces of canonically polarised manifolds which also implies an analogue of Theorem 1.1, [VZ03].

Theorem 2.2 (Hyperbolicity for families of canononically polarized varieties, [Kov00, VZ03])**.** *Let $f^\circ : X^\circ \to Y^\circ$ be a smooth, complex, projective family of canonically polarised varieties of arbitrary dimension, over a smooth quasi-projective base curve Y°. Then the conclusion of the Shafarevich hyperbolicity conjecture, Theorem 1.1, holds.* □

2.2. Families over higher dimensional base manifolds

This paper discusses generalisations of the Shafarevich hyperbolicity conjecture to families over higher dimensional base manifolds. To formulate any generalisation, two points need to be clarified.

(1) We need to define a higher dimensional analogue for the list of quasi-projective curves given in Theorem 1.1.
(2) Given any family $f^\circ : X^\circ \to Y^\circ$ over a higher dimensional base, call two points $y_1,\, y_2 \in Y^\circ$ equivalent if the fibres $(f^\circ)^{-1}(y_1)$ and $(f^\circ)^{-1}(y_2)$ are isomorphic. If Y° is a curve, then either there is only one equivalence class, or all equivalence classes are finite. For families over higher dimensional base manifolds, the equivalence classes will generally be subvarieties of arbitrary dimension. We will need to have a quantitative measure for the number of equivalence classes and their dimensions.

The problems outlined above justify the definition of the *logarithmic Kodaira dimension* and of the *variation of a family*, respectively. Before coming to the generalisations of the Shafarevich hyperbolicity conjecture in Section 2.2.3 below, we recall the definitions for the reader's convenience.

2.2.1. The logarithmic Kodaira dimension The logarithmic Kodaira dimension generalises the classical notion of Kodaira dimension to the category of quasi-projective varieties.

Definition 2.3 (Logarithmic Kodaira dimension)**.** *Let Y° be a smooth quasi-projective variety and Y a smooth projective compactification of Y° such that $D := Y \setminus Y^\circ$ is a divisor with simple normal crossings. The logarithmic Kodaira dimension of Y°, denoted by $\kappa(Y^\circ)$, is defined to be the Kodaira-Iitaka dimension of the line bundle $\mathcal{O}_Y(K_Y + D) \in \operatorname{Pic}(Y)$. A quasi-projective variety Y° is called of* log general type *if $\kappa(Y^\circ) = \dim Y^\circ$, i.e., the divisor $K_Y + D$ is big.*

It is a standard fact of logarithmic geometry that a compactification Y with the described properties exists, and that the logarithmic Kodaira dimension $\kappa(Y^\circ)$ does not depend on the choice of the compactification.

Observation 2.4. The quasi-projective curves listed in Theorem 1.1 are precisely those curves Y° with logarithmic Kodaira dimension $\kappa(Y^\circ) \leq 0$.

2.2.2. The variation of a family The following definition provides a quantitative measure of the *birational* variation of a family. Note that the definition is meaningful even in cases where no moduli space exists.

Definition 2.5 (Variation of a family, cf. [Vie83, Introduction])**.** *Let $f^\circ : X^\circ \to Y^\circ$ be a projective family over an irreducible base Y°, and let $\overline{\mathbb{C}(Y^\circ)}$ denote the algebraic closure of the function field of Y°. The variation of f°, denoted by $\operatorname{Var}(f^\circ)$, is defined as the smallest integer ν for which there exists a subfield K of $\overline{\mathbb{C}(Y^\circ)}$, finitely generated of transcendence degree ν over $\mathbb{C}$ and a K-variety F such that $X \times_{Y^\circ} \operatorname{Spec} \overline{\mathbb{C}(Y^\circ)}$ is birationally equivalent to $F \times_{\operatorname{Spec} K} \operatorname{Spec} \overline{\mathbb{C}(Y^\circ)}$.*

Remark 2.6. In the setup of Definition 2.5, assume that all fibres if Y° are canonically polarised complex manifolds. Then coarse moduli schemes are known to

exist, [Vie95, Thm. 1.11], and the variation is the same as either the dimension of the image of Y° in moduli, or the rank of the Kodaira-Spencer map at the general point of Y°. Further, one obtains that $\mathrm{Var}(f^\circ) = 0$ if and only if all fibres of f° are isomorphic. In this case, the family f° is called "isotrivial".

2.2.3. Viehweg's conjecture Using the notion of "logarithmic Kodaira dimension" and "variation", the Shafarevich hyperbolicity conjecture can be reformulated as follows.

Theorem 2.7 (Reformulation of Theorem 1.1)**.** *If $f^\circ : X^\circ \to Y^\circ$ is any smooth, complex, projective family of curves of genus $g > 1$, over a smooth quasi-projective base curve Y°, and if $\mathrm{Var}(f^\circ) = \dim Y^\circ$, then $\kappa(Y^\circ) = \dim Y^\circ$.*

Aiming to generalise the Shafarevich hyperbolicity conjecture to families over higher dimensional base manifolds, Viehweg has conjectured that this reformulation holds true in arbitrary dimension.

Conjecture 2.8 (Viehweg's conjecture, [Vie01, 6.3])**.** *Let $f^\circ : X^\circ \to Y^\circ$ be a smooth projective family of varieties with semiample canonical bundle, over a quasi-projective manifold Y°. If f° has maximal variation, then Y° is of log general type. In other words,*

$$\mathrm{Var}(f^\circ) = \dim Y^\circ \;\Rightarrow\; \kappa(Y^\circ) = \dim Y^\circ.$$

Viehweg's conjecture has been proven by Sándor Kovács and the author in case where Y° is a surface, [KK08a, KK07], or a threefold, [KK08c]. The methods developed in these papers will be discussed, and an idea of the proof will be given later in this paper, cf. the outline of this paper given in Section 1.2 on page 3.

Theorem 2.9 (Viehweg's conjecture for families over threefolds, [KK08c, Thm. 1.1])**.** *Viehweg's conjecture holds in case where $\dim Y^\circ \leq 3$.* □

For families of *canonically polarised* varieties, much stronger results have been obtained, giving an explicit geometric explanation of Theorem 2.9.

Theorem 2.10 (Relationship between the moduli map and the MMP, [KK08c, Thm. 1.1])**.** *Let $f^\circ : X^\circ \to Y^\circ$ be a smooth projective family of canonically polarised varieties, over a quasi-projective manifold Y° of dimension $\dim Y^\circ \leq 3$. Let Y be a smooth compactification of Y° such that $D := Y \setminus Y^\circ$ is a divisor with simple normal crossings.*

Then any run of the minimal model program of the pair (Y, D) will terminate in a Kodaira or Mori fibre space whose fibration factors the moduli map birationally.

Remark 2.11. Neither the compactification Y nor the minimal model program discussed in Theorem 2.10 is unique. When running the minimal model program, one often needs to choose the extremal ray that is to be contracted.

In order to explain the statement of Theorem 2.10, let $\mathfrak{M}$ be the appropriate coarse moduli space whose existence is shown, e.g. in [Vie95, Thm. 1.11]. Further, let $\mu^\circ : Y^\circ \to \mathfrak{M}$ be the moduli map associated with the family f°, and let $\mu : Y \dashrightarrow \mathfrak{M}$ be the associated rational map from the compactification Y. If $\lambda : Y \dashrightarrow Y_\lambda$ is a rational map obtained by running the minimal model program, and if $Y_\lambda \to Z_\lambda$ is the associated Kodaira or Mori fibre space, then Theorem 2.10 asserts the existence of a map $Z_\lambda \dashrightarrow \mathfrak{M}$ that makes the following diagram commutative,

$$\begin{tikzcd} Y \arrow[rrr, dashed, "\lambda", "\text{MMP of the pair } (Y,D)"'] \arrow[d, dashed, "\text{moduli map induced by } f^\circ"'] & & & Y_\lambda \arrow[d, "\text{Kodaira or Mori fibre space}"] \\ \mathfrak{M} & & & Z_\lambda . \arrow[lll, dashed, "\exists !"] \end{tikzcd}$$

Now, if we assume in addition that $\kappa(Y^\circ) \geq 0$, then the minimal model program terminates in a Kodaira fibre space whose base Z_λ has dimension $\dim Z_\lambda = \kappa(Y^\circ)$, so that $\operatorname{Var}(f^\circ) \leq \kappa(Y^\circ)$. If we assume that $\kappa(Y^\circ) = -\infty$, then the minimal model program terminates in proper Mori fibre space and we obtain that $\dim Z_\lambda < \dim Y$ and $\operatorname{Var}(f^\circ) < \dim Y^\circ$. The following refined answer to Viehweg's conjecture is therefore an immediate corollary of Theorem 2.10.

Corollary 2.12 (Refined answer to Viehweg's conjecture, [KK08c, Cor. 1.3])**.** *Let $f^\circ : X^\circ \to Y^\circ$ be a smooth projective family of canonically polarised varieties, over a quasi-projective manifold Y° of dimension $\dim Y^\circ \leq 3$. Then either*

(1) $\kappa(Y^\circ) = -\infty$ *and* $\operatorname{Var}(f^\circ) < \dim Y^\circ$, *or*
(2) $\kappa(Y^\circ) \geq 0$ *and* $\operatorname{Var}(f^\circ) \leq \kappa(Y^\circ)$. □

Remark 2.13. Corollary 2.12 asserts that any family of canonically polarised varieties over a base manifold Y° with $\kappa(Y^\circ) = 0$ is necessarily isotrivial.

Remark 2.14. Corollary 2.12 has also been shown in case where Y° is a *projective* manifold of arbitrary dimension, conditional to the standard conjectures of minimal model theory[1], cf. [KK08b, Thm. 1.4]. A very short proof that does not rely on minimal model theory has been announced by Patakfalvi as this paper goes to print, [Pat11].

Example 2.15 (Optimality of Corollary 2.12 in case $\kappa(Y^\circ) = -\infty$). To see that the result of Corollary 2.12 is optimal in case $\kappa(Y^\circ) = -\infty$, let $f_1^\circ : X_1^\circ \to Y_1^\circ$ be any family of canonically polarised varieties with $\operatorname{Var}(f_1^\circ) = 2$, over a smooth surface Y_1° (which may or may not be compact). Setting $X^\circ := X_1^\circ \times \mathbb{P}^1$ and $Y^\circ := Y_1^\circ \times \mathbb{P}^1$, we obtain a family $f^\circ = f_1^\circ \times \mathrm{Id}_{\mathbb{P}^1} : X^\circ \to Y^\circ$ with variation $\operatorname{Var}(f^\circ) = 2$, and with a base manifold Y° of Kodaira dimension $\kappa(Y^\circ) = -\infty$.

[1] i.e., existence and termination of the minimal model program and abundance

Example 2.16 (Related and complementary results in case $\kappa(Y^\circ) = -\infty$). In the setup of Corollary 2.12, if Y° is a projective Fano manifold, then a fundamental result of Campana and Kollár-Miyaoka-Mori asserts that Y° is rationally connected, [Kol96, V. Thm. 2.13]. In other words, given any two points x, y in Y°, there exists a rational curve $C \subset Y^\circ$ which contains both x and y. Recalling from Theorem 2.2 that families over rational curves are isotrivial, it follows immediately that the family f° is necessarily isotrivial itself.

A much stronger version of this result has been shown by Lohmann, [Loh11]. Given a projective variety Y and a $\mathbb{Q}$-divisor D such that (Y, D) is a divisorially log terminal (=dlt)[2] pair, consider the smooth quasi-projective variety

$$Y^\circ := (Y \setminus \operatorname{supp}\lfloor D \rfloor)_{\mathrm{reg}}.$$

Lohmann shows that if (Y, D) is log-Fano, that is, if the $\mathbb{Q}$-divisor $-(K_Y + D)$ is ample, then any family of canonically polarized varieties over Y° is necessarily isotrivial. The proof relies on a generalization of Araujo's result [Ara10] which relates extremal rays in the moving cone of a variety with fiber spaces that appear at the end of the minimal model program. Lohmann shows that the moduli map factorizes through any of the fibrations obtained in this way.

2.2.4. Campana's conjecture In a series of papers, including [Cam04, Cam08], Campana introduced the notion of "geometric orbifolds" and "special varieties". Campana's language helps to formulate a very natural generalisation of Theorem 1.1, which includes the cases covered by the Viehweg Conjecture 2.8, and gives (at least conjecturally) a satisfactory geometric explanation of isotriviality observed in some families over spaces that are not covered by Conjecture 2.8.

Before formulating the conjecture, we briefly recall the precise definition of a special logarithmic pair for the reader's convenience. We take the classical Bogomolov-Sommese Vanishing Theorem as our starting point. We refer to [Iit82, EV92] or to the original reference [Del70] for an explanation of the sheaf $\Omega^p_Y(\log D)$ of logarithmic differentials.

Theorem 2.17 (Bogomolov-Sommese Vanishing Theorem, cf. [EV92, Sect. 6])**.** *Let Y be a smooth projective variety and $D \subset Y$ a reduced (possibly empty) divisor with simple normal crossings. If $p \leq \dim Y$ is any number and $\mathcal{A} \subseteq \Omega^p_Y(\log D)$ any invertible subsheaf, then the Kodaira-Iitaka dimension of $\mathcal{A}$ is at most p, i.e., $\kappa(\mathcal{A}) \leq p$.* □

In a nutshell, we say that a pair (Y, D) is special if the inequality in the Bogomolov-Sommese Vanishing Theorem is always strict.

Definition 2.18 (Special logarithmic pair)**.** *In the setup of Theorem 2.17, a pair (Y, D) is called* special *if the strict inequality $\kappa(\mathcal{A}) < p$ holds for all p and all*

[2]We refer to [KM98, Sect. 2.3] for the definition of a *dlt* pair, and for related notions concerning singularities of pairs that are relevant in minimal model theory.

invertible sheaves $\mathcal{A} \subseteq \Omega^p_Y(\log D)$. A smooth, quasi-projective variety Y° is called special if there exists a smooth compactification Y such that $D := Y \setminus Y^\circ$ is a divisor with simple normal crossings and such that the pair (Y, D) is special.

Remark 2.19 (Special quasi-projective variety). It is an elementary fact that if Y° is a smooth, quasi-projective variety and Y_1, Y_2 two smooth compactifications such that $D_i := Y_i \setminus Y^\circ$ are divisors with simple normal crossings, then (Y_1, D_1) is special if and only if (Y_2, D_2) is. The notion of special should thus be seen as a property of the quasi-projective variety Y°.

Fact 2.20 (Examples of special manifolds, cf. [Cam04, Thms. 3.22 and 5.1]). Rationally connected manifolds and manifolds X with $\kappa(X) = 0$ are special. □

With this notation in place, Campana's conjecture can be formulated as follows.

Conjecture 2.21 (Campana's conjecture, [Cam08, Conj. 12.19]). *Let $f : X^\circ \to Y^\circ$ be a smooth family of canonically polarised varieties over a smooth quasi-projective base. If Y° is special, then the family f is isotrivial.*

In analogy with the construction of the maximally rationally connected quotient map of uniruled varieties, Campana constructs in [Cam04, Sect. 3] an almost-holomorphic "core map" whose fibres are special in the sense of Definition 2.18. Like the MRC quotient, the core map is uniquely characterised by certain maximality properties, [Cam04, Thm. 3.3], which essentially say that the core map of X contracts almost all special subvarieties contained in X. If Campana's Conjecture 2.21 holds, this would imply that the core map always factors the moduli map, similar to what we have seen in Section 2.2.3 above,

$$\begin{tikzcd}
Y \arrow[rr, dashed, "\text{core map}", "\text{almost holomorphic}"'] \arrow[d, dashed, "\text{moduli map induced by } f^\circ"'] & & Z \arrow[dll, dashed, bend left, "\exists !"] \\
\mathfrak{M}. & &
\end{tikzcd}$$

As with Viehweg's Conjecture 2.8, Campana's Conjecture 2.21 has been shown for surfaces [JK09b] and threefolds [JK09a].

Theorem 2.22 (Campana's conjecture in dimension three, [JK09a, Thm. 1.5]). *Campana's Conjecture 2.21 holds if* $\dim Y^\circ \leq 3$. □

2.3. Conjectures and open problems

Viehweg's Conjecture 2.8 and Campana's Conjecture 2.21 have been shown for families over base manifolds of dimension three or less. As we will see in Section 5, the restriction to three-dimensional base manifolds comes from the fact that minimal model theory is particularly well-developed for threefolds, and from our limited ability to handle differential forms on singular spaces of higher

dimension. We do not believe that there is a fundamental reason that restricts us to dimension three, and we do believe that the relationship between the moduli map and the MMP found in Theorem 2.10 will hold in arbitrary dimension.

Conjecture 2.23 (Relationship between the moduli map and the MMP)**.** *Let $f^\circ : X^\circ \to Y^\circ$ be a smooth projective family of canonically polarised varieties, over a quasi-projective manifold Y°. Let Y be a smooth compactification of Y° such that $D := Y \setminus Y^\circ$ is a divisor with simple normal crossings. Then any run of the minimal model program of the pair (Y, D) will terminate in a Kodaira or Mori fibre space whose fibration factors the moduli map birationally.*

Conjecture 2.24 (Refined Viehweg conjecture, cf. [KK08a, Conj. 1.6])**.** *Corollary 2.12 holds without the assumption that* $\dim Y^\circ \leq 3$.

Given the current progress in minimal model theory, a proof of Conjectures 2.23 and 2.24 does no longer seem out of reach.

3. Techniques I: Existence of Pluri-differentials on the base of a family

3.1. The existence result

Throughout the present Section 3, we consider a smooth projective family $f^\circ : X^\circ \to Y^\circ$ of projective, canonically polarised complex manifolds, over a smooth complex quasi-projective base. We assume that the family is not isotrivial, and fix a smooth projective compactification Y of Y° such that $D := Y \setminus Y^\circ$ is a divisor with simple normal crossings. In this setup, Viehweg and Zuo have shown the following fundamental result asserting the existence of many logarithmic pluri-differentials on Y.

Theorem 3.1 (Existence of pluri-differentials on Y, [VZ02, Thm. 1.4(i)])**.** *Let $f^\circ : X^\circ \to Y^\circ$ be a smooth projective family of canonically polarised complex manifolds, over a smooth complex quasi-projective base. Assume that the family is not isotrivial and fix a smooth projective compactification Y of Y° such that $D := Y \setminus Y^\circ$ is a divisor with simple normal crossings.*

Then there exists a number $m > 0$ and an invertible sheaf $\mathcal{A} \subseteq \operatorname{Sym}^m \Omega^1_Y(\log D)$ whose Kodaira-Iitaka dimension is at least the variation of the family, $\kappa(\mathcal{A}) \geq \operatorname{Var}(f^\circ)$. □

Remark 3.2. Observe that the Shafarevich hyperbolicity conjecture, Theorem 1.1, follows as an immediate corollary of Theorem 3.1.

Remark 3.3. A somewhat weaker version of Theorem 3.1 holds for families of projective manifolds with only semiample canonical bundle if one assumes additionally that the family is of maximal variation, i.e., that $\operatorname{Var}(f^\circ) = \dim Y^\circ$, cf. [VZ02, Thm. 1.4(iv)].

As we will see in Section 5, the "Viehweg-Zuo" sheaf $\mathcal{A}$ is one of the crucial ingredients in the proofs of Viehweg's and Campana's conjecture for families over threefolds, Theorems 2.9, 2.10 and 2.22. A careful review of Viehweg and Zuo's construction reveals that the "Viehweg-Zuo sheaf" $\mathcal{A}$ comes from the coarse moduli space $\mathfrak{M}$, at least generically. The precise statement, given in Theorem 3.6, uses the following notion.

Notation 3.4 (Differentials coming from moduli space generically)**.** *Let $\mu : Y^\circ \to \mathfrak{M}$ be the moduli map associated with the family f°, and consider the subsheaf $\mathcal{B} \subseteq \Omega^1_Y(\log D)$, defined on presheaf level as follows: if $U \subseteq Y$ is any open set and $\sigma \in H^0\big(U, \Omega^1_Y(\log D)\big)$ any section, then $\sigma \in H^0\big(U, \mathcal{B}\big)$ if and only if the restriction $\sigma|_{U'}$ is in the image of the differential map*

$$d\mu|_{U'} : \mu^*\big(\Omega^1_{\mathfrak{M}}\big)|_{U'} \longrightarrow \Omega^1_{U'},$$

where $U' \subseteq U \cap Y^\circ$ is the open subset where the moduli map μ has maximal rank.

Remark 3.5. By construction, it is clear that the sheaf $\mathcal{B}$ is a saturated subsheaf of $\Omega^1_Y(\log D)$, i.e., that the quotient sheaf $\Omega^1_Y(\log D)/\mathcal{B}$ is torsion free. We say that $\mathcal{B}$ is the saturation of $\mathrm{Image}(d\mu)$ in $\Omega^1_Y(\log D)$.

Theorem 3.6 (Refinement of the Viehweg-Zuo Theorem 3.1, [JK09b, Thm. 1.4])**.** *In the setup of Theorem 3.1, there exists a number $m > 0$ and an invertible subsheaf $\mathcal{A} \subseteq \mathrm{Sym}^m \mathcal{B}$ whose Kodaira-Iitaka dimension is at least the variation of the family, $\kappa(\mathcal{A}) \geq \mathrm{Var}(f^\circ)$.*

Theorem 3.6 follows without too much work from Viehweg's and Zuo's original arguments and constructions, which are reviewed in Section 3.2 below. Compared with Theorem 3.1, the refined Viehweg-Zuo theorem relates more directly to Campana's Conjecture 2.21 and other generalizations of the Shafarevich conjecture. To illustrate its use, we show in the surface case how Theorem 3.6 reduces Campana's Conjecture 2.21 to the Viehweg Conjecture 2.8, for which a positive answer is known.

Corollary 3.7 (Campana's conjecture in dimension two)**.** *Conjecture 2.21 holds if* $\dim Y^\circ = 2$.

Proof. We maintain the notation of Conjecture 2.21 and let $f : X^\circ \to Y^\circ$ be a smooth family of canonically polarised varieties over a smooth quasi-projective base, with Y° a special surface. Since Y° is special, it is not of log general type, and the solution to Viehweg's conjecture in dimension two, [KK08c, Thm. 1.1], gives that $\mathrm{Var}(f^\circ) < 2$.

We argue by contradiction, suppose that $\mathrm{Var}(f^\circ) = 1$ and choose a compactification (Y, D) as in Definition 2.18. By Theorem 3.6 there exists a number $m > 0$ and an invertible subsheaf $\mathcal{A} \subseteq \mathrm{Sym}^m \mathcal{B}$ such that $\kappa(\mathcal{A}) \geq 1$. However, since $\mathcal{B}$ is saturated in the locally free sheaf $\Omega^1_Y(\log D)$, it is reflexive, [OSS80, Claim on

p. 158], and since $\operatorname{Var}(f^\circ) = 1$, the sheaf $\mathcal{B}$ is of rank 1. Thus $\mathcal{B} \subseteq \Omega^1_Y(\log D)$ is an invertible subsheaf, [OSS80, Lem. 1.1.15, on p. 154], and Definition 2.18 of a special pair gives that $\kappa(\mathcal{B}) < 1$, contradicting the fact that $\kappa(\mathcal{A}) \geq 1$. It follows that $\operatorname{Var}(f^\circ) = 0$ and that the family is hence isotrivial. □

Outline of this section Given its importance in the theory, we give a very brief synopsis of Viehweg-Zuo's proof of Theorem 3.1, showing how the theorem follows from deep positivity results[3] for push-forward sheaves of relative dualizing sheaves, and for kernels of Kodaira-Spencer maps, respectively. Even though no proof of the refined Theorem 3.6, is given, it is hoped that the reader who chooses to read Section 3.2 will believe that Theorem 3.6 follows with some extra work by essentially the same methods.

The reader who is interested in a detailed understanding, including is referred to the papers [Kol86], [VZ02], and to the survey [Vie01]. The overview contained in this section and the facts outlined in Section 3.2.5 can perhaps serve as a guideline to [VZ02].

Many of the technical difficulties encountered in the full proof of Theorem 3.1 vanish if f° is a family of curves. The proof becomes very transparent in this case. In particular, it is very easy to see how the Kodaira-Spencer map associated with the family f° transports the positivity found in push-forward sheaves into the sheaf of differentials $\Omega^1_Y(\log Y)$. After setting up notation in Section 3.2.1, we have therefore included a Section 3.2.2 which discusses the curve case in detail.

Most of the material presented in the current Section 3, including the synopsis of Viehweg-Zuo's construction, is taken without much modification from the paper [JK09b]. The presentation is inspired in part by [Vie01].

3.2. A synopsis of Viehweg-Zuo's construction

3.2.1. Setup of notation Throughout the present Section 3.2, we choose a smooth projective compactification X of X° such that the following holds:

(1) The difference $\Delta := X \setminus X^\circ$ is a divisor with simple normal crossings.
(2) The morphism f° extends to a projective morphism $f : X \to Y$.

It is then clear that $\Delta = f^{-1}(D)$ set-theoretically. Removing a suitable subset $S \subset Y$ of codimension $\operatorname{codim}_Y S \geq 2$, the following will then hold automatically on $Y' := Y \setminus S$ and $X' := X \setminus f^{-1}(S)$, respectively.

(3) The restricted morphism $f' := f|_{X'}$ is flat.
(4) The divisor $D' := D \cap Y'$ is smooth.
(5) The divisor $\Delta' := \Delta \cap X'$ is a relative normal crossing divisor, i.e. a normal crossing divisor whose components and all their intersections are smooth over the components of D'.

[3] The positivity results in question are formulated in Theorems 3.10 and Fact 3.22, respectively.

In the language of Viehweg-Zuo, [VZ02, Def 2.1(c)], the restricted morphism $f' : X' \to Y'$ is a "good partial compactification of f°".

Remark 3.8 (Restriction to a partial compactification)*.* Let $\mathcal{G}$ be a locally free sheaf on Y, and let $\mathcal{F}' \subseteq \mathcal{G}|_{Y'}$ be an invertible subsheaf. Since $\operatorname{codim}_Y S \geq 2$, there exists a unique extension of the sheaf $\mathcal{F}'$ to an invertible subsheaf $\mathcal{F} \subseteq \mathcal{G}$ on Y. Furthermore, the restriction map $H^0\big(Y, \mathcal{F}\big) \to H^0\big(Y', \mathcal{F}'\big)$ is an isomorphism. In particular, the notion of Kodaira-Iitaka dimension makes sense for the sheaf $\mathcal{F}'$, and $\kappa(\mathcal{F}') = \kappa(\mathcal{F})$.

We denote the relative dimension of X over Y by $n := \dim X - \dim Y$.

3.2.2. Idea of proof of Theorem 3.1 for families of curves Before sketching the proof of Theorem 3.1 in full generality, we illustrate the main idea in a particularly simple setting.

Simplifying Assumptions 3.9*.* Throughout the present introductory Section 3.2.2, we maintain the following simplifying assumptions in addition to the assumptions made in Theorem 3.1.

(1) The quasi-projective variety Y° is in fact projective. In particular, we assume that $X = X^\circ$, $f = f^\circ$, that $D = \emptyset$ and that $\Delta = \emptyset$.
(2) The family $f : X \to Y$ is a family of curves of genus $g > 1$. In particular, we have that $(T_{X/Y})^* = \Omega^1_{X/Y} = \omega_{X/Y}$, where $T_{X/Y}$ is the kernel of the derivative $Tf : T_X \to f^*(T_Y)$.
(3) The variation of f° is maximal, that is, $\operatorname{Var}(f^\circ) = \dim Y^\circ$.

The proof of Theorem 3.1 sketched here uses positivity of the push-forward of relative dualizing sheaves as its main input. The positivity result required is discussed in Viehweg's survey [Vie01, Sect. 1–3], where positivity is obtained as a consequence of generalised Kodaira vanishing theorems. The reader interested in a broader overview might also want to look at the remarks and references in [Laz04, Sect. 6.3.E], as well as the papers [Kol86, Zuo00]

Theorem 3.10 (Positivity of push-forward sheaves, cf. [VZ02, Prop. 3.4.(i)])**.** *Under the simplifying Assumptions 3.9, the push-forward sheaf $f_*\big(\omega_{X/Y}^{\otimes 2}\big)$ is locally free of positive rank. If $\mathcal{A} \in \operatorname{Pic}(Y)$ is any ample line bundle, then there exist numbers $N, M \gg 0$ and a sheaf morphism*

$$\phi : \mathcal{A}^{\oplus M} \to \operatorname{Sym}^N f_*\big(\omega_{X/Y}^{\otimes 2}\big)$$

which is surjective at the general point of Y. □

For the reader's convenience, we recall two other facts used in the proof, namely the existence of a Kodaira-Spencer map, and Serre duality in the relative setting.

Theorem 3.11 (Kodaira-Spencer map, cf. [Voi07, Sect. 9.1.2] or [Huy05, Sect. 6.2])**.** *Under the simplifying Assumptions 3.9, since* $\mathrm{Var}(f) > 0$*, there exists a non-zero sheaf morphism* $\kappa : T_Y \to R^1 f_*(T_{X/Y})$ *which measures the variation of the isomorphism classes of fibres in moduli.* □

Theorem 3.12 (Serre duality in the relative setting, cf. [Liu02, Sect. 6.4])**.** *Under the simplifying Assumptions 3.9, if* $\mathcal{F}$ *is any coherent sheaf on* X*, then there exists a natural isomorphism* $f_*(\mathcal{F}^* \otimes \omega_{X/Y}) \cong \big(R^1 f_*(\mathcal{F})\big)^*$. □

Proof of Theorem 3.1 under the Simplifying Assumptions 3.9. Consider the dual of the (non-trivial) Kodaira-Spencer map discussed in Theorem 3.11, say $\kappa^* : \big(R^1 f_*(T_{X/Y})\big)^* \to (T_Y)^*$. Recalling that $T^*_{X/Y}$ equals the relative dualizing sheaf $\omega_{X/Y}$, and using the relative Serre Duality Theorem 3.12, the sheaf morphism κ^* is naturally identified with a non-zero morphism

$$\kappa^* : f_*\big(\omega^{\otimes 2}_{X/Y}\big) \to \Omega^1_Y. \tag{3.13}$$

Choosing an ample line bundle $\mathcal{A} \in \mathrm{Pic}(Y)$ and sufficiently large and divisible numbers $N, M \gg 0$, Theorem 3.10 yields a sequence of sheaf morphisms

$$\mathcal{A}^{\oplus M} \xrightarrow[\text{gen. surjective}]{\phi} \mathrm{Sym}^N f_*\big(\omega^{\otimes 2}_{X/Y}\big) \xrightarrow[\text{non-trivial}]{\mathrm{Sym}^N(\kappa^*)} \mathrm{Sym}^N \Omega^1_Y,$$

whose composition $\mathcal{A}^{\oplus M} \to \mathrm{Sym}^N \Omega^1_Y$ is clearly not the zero map. Consequently, we obtain a non-trivial map $\mathcal{A} \to \mathrm{Sym}^N \Omega^1_Y$, finishing the proof of Theorem 3.1 under the Simplifying Assumptions 3.9. □

The proof outlined above uses the dual of the Kodaira-Spencer map as a vehicle to transport the positivity which exists in $f_*\big(\omega^{\otimes 2}_{X/Y}\big)$ into the sheaf Ω^1_Y of differential forms on Y. If f was a family of surfaces rather than a family of curves, then Serre duality cannot easily be used to identify the dual of $R^1 f_*(T_{X/Y})$ with a push-forward sheaf of type $f_*\big(\omega^{\otimes \bullet}_{X/Y}\big)$, or any with other sheaf whose positivity is well-known. To overcome this problem, Viehweg suggested to replace the Kodaira-Spencer map κ by sequences of more complicated sheaf morphisms $\tau^0_{p,q}$ and τ^k, constructed in Sections 3.2.3 and 3.2.4 below. To motivate the slightly involved construction of these maps, we recall without proof a description of the classical Kodaira-Spencer map.

Construction 3.14. Under the Simplifying Assumptions 3.9, consider the standard sequence of relative differential forms on X,

$$0 \to f^*\Omega^1_Y \to \Omega^1_X \to \Omega^1_{X/Y} \to 0, \tag{3.15}$$

and its twist with the invertible sheaf $\omega^*_{X/Y}$,

$$0 \to f^*\Omega^1_Y \otimes \omega^*_{X/Y} \to \Omega^1_X \otimes \omega^*_{X/Y} \to \underbrace{\Omega^1_{X/Y} \otimes \omega^*_{X/Y}}_{\cong \mathcal{O}_X} \to 0.$$

Using that $f_*(\mathcal{O}_X) = \mathcal{O}_Y$, the first connecting morphism associated with this sequence then reads

$$\mathcal{O}_Y \to \Omega^1_Y \otimes R^1 f_*(\omega^*_{X/Y}) =: \mathcal{F}. \tag{3.16}$$

The sheaf $\mathcal{F}$ is naturally isomorphic to the sheaf $\operatorname{Hom}\big(T_Y, R^1 f_*(T_{X/Y})\big)$. To give a morphism $\mathcal{O}_Y \to \mathcal{F}$ is thus the same as to give a map $T_Y \to R^1 f_*(T_{X/Y})$, and the morphism obtained in (3.16) is the same as the Kodaira-Spencer map discussed in Theorem 3.11.

Observe also that Serre duality yields a natural identification of $\mathcal{F}$ with the sheaf $\operatorname{Hom}\big(f_*(\omega^{\otimes 2}_{X/Y}), \Omega^1_Y\big)$. To give a morphism $\mathcal{O}_Y \to \mathcal{F}$ it is thus the same as to give a map $f_*(\omega^{\otimes 2}_{X/Y}) \to \Omega^1_Y$. The morphism obtained in this way from (3.16) is of course the morphism κ^* of Equation (3.13).

3.2.3. Proof of Theorem 3.1, construction of the $\boldsymbol{\tau^0_{p,q}}$ In the general setting of Theorem 3.1 where the simplifying Assumptions 3.9 do not generally hold, the starting point of the Viehweg-Zuo construction is the standard sequence of relative logarithmic differentials associated to the flat morphism f' which generalises Sequence (3.15) from above,

$$0 \to (f')^*\Omega^1_{Y'}(\log D') \to \Omega^1_{X'}(\log \Delta') \to \Omega^1_{X'/Y'}(\log \Delta') \to 0. \tag{3.17}$$

We refer to [EV90, Sect. 4] for a discussion of Sequence (3.17), and for a proof of the fact that the cokernel $\Omega^1_{X'/Y'}(\log \Delta')$ is locally free. By [Har77, II, Ex. 5.16], Sequence (3.17) defines a filtration of the p^{th} exterior power,

$$\Omega^p_{X'}(\log \Delta') = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^p \supseteq F^{p+1} = \{0\},$$

with $F^r/F^{r+1} \cong (f')^*\big(\Omega^r_{Y'}(\log D')\big) \otimes \Omega^{p-r}_{X'/Y'}(\log \Delta')$. Take the first sequence induced by the filtration,

$$0 \longrightarrow F^1 \longrightarrow F^0 \longrightarrow F^0/F^1 \longrightarrow 0,$$

modulo F^2, and obtain

$$\begin{multline} 0 \longrightarrow (f')^*\big(\Omega^1_{Y'}(\log D')\big) \otimes \Omega^{p-1}_{X'/Y'}(\log \Delta') \longrightarrow F^0/F^2 \longrightarrow \\ \longrightarrow \Omega^p_{X'/Y'}(\log \Delta') \longrightarrow 0. \end{multline} \tag{3.18}$$

Setting $\mathcal{L} := \Omega^n_{X'/Y'}(\log \Delta')$, twisting Sequence (3.18) with $\mathcal{L}^{-1}$ and pushing down, the connecting morphisms of the associated long exact sequence give maps

$$\tau^0_{p,q} : F^{p,q} \longrightarrow F^{p-1,q+1} \otimes \Omega^1_{Y'}(\log D'),$$

where $F^{p,q} := R^q f'_*(\Omega^p_{X'/Y'}(\log \Delta') \otimes \mathcal{L}^{-1})/\text{torsion}$. Set $\mathcal{N}^{p,q}_0 := \ker(\tau^0_{p,q})$.

3.2.4. Alignment of the $\tau^0_{p,q}$ The morphisms $\tau^0_{p,q}$ and $\tau^0_{p-1,q+1}$ can be composed if we tensor the latter with the identity morphism on $\Omega^1_{Y'}(\log D')$. More specifically, we consider the following morphisms,

$$\underbrace{\tau^0_{p,q} \otimes \mathrm{Id}_{\Omega^1_{Y'}(\log D')^{\otimes q}}}_{=:\tau_{p,q}} : F^{p,q} \otimes \big(\Omega^1_{Y'}(\log D')\big)^{\otimes q} \to F^{p-1,q+1} \otimes \big(\Omega^1_{Y'}(\log D')\big)^{\otimes q+1},$$

and their compositions

$$\underbrace{\tau_{n-k+1,k-1} \circ \cdots \circ \tau_{n-1,1} \circ \tau_{n,0}}_{=:\tau^k} : F^{n,0} \to F^{n-k,k} \otimes \big(\Omega^1_{Y'}(\log D')\big)^{\otimes k}. \tag{3.19}$$

3.2.5. Fundamental facts about τ^k and $\mathcal{N}^{p,q}_0$ Theorem 3.1 is shown by relating the morphism $\tau^0_{p,q}$ with the structure morphism of a Higgs-bundle coming from the variation of Hodge structures associated with the family f°. Viehweg's positivity results of push-forward sheaves of relative differentials, as well as Zuo's results on the curvature of kernels of generalised Kodaira-Spencer maps are the main input here. Rather than recalling the complicated line of argumentation, we simply state two central results from the argumentation of [VZ02].

Fact 3.20 (Factorization via symmetric differentials, [VZ02, Lem. 4.6]). For any k, the morphism τ^k factors via the symmetric differentials $\mathrm{Sym}^k \Omega^1_{Y'}(\log D') \subseteq \big(\Omega^1_{Y'}(\log D')\big)^{\otimes k}$. More precisely, the morphism τ^k takes its image in $F^{n-k,k} \otimes \mathrm{Sym}^k \Omega^1_{Y'}(\log D')$. □

Consequence 3.21. Using Fact 3.20 and the observation that $F^{n,0} \cong \mathcal{O}_{Y'}$, we can therefore view τ^k as a morphism

$$\tau^k : \mathcal{O}_{Y'} \longrightarrow F^{n-k,k} \otimes \mathrm{Sym}^k \Omega^1_{Y'}(\log D').$$

While the proof of Fact 3.20 is rather elementary, the following deep result is at the core of Viehweg-Zuo's argument. Its role in the proof of Theorem 3.1 is comparable to that of the Positivity Theorem 3.10 discussed in Section 3.2.2.

Fact 3.22 (Negativity of $\mathcal{N}^{p,q}_0$, [VZ02, Claim 4.8]). Given any numbers p and q, there exists a number k and an invertible sheaf $\mathcal{A}' \in \mathrm{Pic}(Y')$ of Kodaira-Iitaka dimension $\kappa(\mathcal{A}') \geq \mathrm{Var}(f^0)$ such that $(\mathcal{A}')^* \otimes \mathrm{Sym}^k\big((\mathcal{N}^{p,q}_0)^*\big)$ is generically generated. □

3.2.6. End of proof To end the sketch of proof, we follow [VZ02, p. 315] almost verbatim. By Fact 3.22, the trivial sheaf $F^{n,0} \cong \mathcal{O}_{Y'}$ cannot lie in the kernel $\mathcal{N}^{n,0}_0$ of $\tau^1 = \tau^0_{n,0}$. We can therefore set $1 \leq m$ to be the largest number with $\tau^m(F^{n,0}) \neq \{0\}$. Since m is maximal, $\tau^{m+1} = \tau_{n-m,m} \circ \tau^m \equiv 0$ and

$$\mathrm{Image}(\tau^m) \subseteq \ker(\tau_{n-m,m}) = \mathcal{N}^{n-m,m}_0 \otimes \mathrm{Sym}^m \Omega^1_{Y'}(\log D').$$

In other words, τ^m gives a non-trivial map

$$\tau^m : \mathcal{O}_{Y'} \cong F^{n,0} \longrightarrow \mathcal{N}^{n-m,m}_0 \otimes \mathrm{Sym}^m \Omega^1_{Y'}(\log D').$$

Equivalently, we can view τ^m as a non-trivial map

$$\tau^m : (\mathcal{N}_0^{n-m,m})^* \longrightarrow \operatorname{Sym}^m \Omega^1_{Y'}(\log D'). \tag{3.23}$$

By Fact 3.22, there are many morphisms $\mathcal{A}' \to \operatorname{Sym}^k\big((\mathcal{N}_0^{n-m,m})^*\big)$, for k large enough. Together with (3.23), this gives a non-zero morphism $\mathcal{A}' \to \operatorname{Sym}^{k\cdot m} \Omega^1_{Y'}(\log D')$.

We have seen in Remark 3.8 that the sheaf $\mathcal{A}' \subseteq \operatorname{Sym}^{k\cdot m} \Omega^1_{Y'}(\log D')$ extends to a sheaf $\mathcal{A} \subseteq \operatorname{Sym}^{k\cdot m} \Omega^1_{Y}(\log D)$ with $\kappa(\mathcal{A}) = \kappa(\mathcal{A}') \geq \operatorname{Var}(f^\circ)$. This ends the proof of Theorem 3.1. $\square$

3.3. Open problems

In spite of its importance, little is known about further properties that the Viehweg-Zuo sheaves $\mathcal{A}$ might have.

Question 3.24. For families of higher-dimensional manifolds, how does the Viehweg-Zuo construction behave under base change? Does it satisfy any universal properties at all? If not, is there a "natural" positivity result for base spaces of families that does satisfy good functorial properties?

In the setup of Theorem 3.1, if $Z^\circ \subset Y^\circ$ is any closed submanifold, then the associated Viehweg-Zuo sheaves $\mathcal{A}$, constructed for the family $f^\circ : X^\circ \to Y^\circ$, and $\mathcal{A}_Z$, constructed for the restricted family $f^\circ_Z : X^\circ \times_{Y^\circ} Z^\circ \to Z^\circ$, may differ. In particular, it is not clear that $\mathcal{A}_Z$ is the restriction of $\mathcal{A}$, and the sheaves $\mathcal{A}$ and $\mathcal{A}_Z$ may live in different symmetric products of their respective Ω^1's.

One likely source of non-compatibility with base change is the choice of the number m in Section 3.2.6 ("largest number with $\tau^m(F^{n,0}) \neq \{0\}$"). It seems unlikely that this definition behaves well under base change.

Question 3.25. For families of higher-dimensional manifolds, are there distinguished subvarieties in moduli space that have special Viehweg-Zuo sheaves, perhaps contained in particularly high/low symmetric powers of Ω^1? Does the lack of a restriction morphism induce a geometric structure on the moduli space?

The refinement of the Viehweg-Zuo Theorem, presented in Theorem 3.6 above, turns out to be important for the applications that we have in mind. It is, however, not clear to us if the sheaf $\mathcal{B}$ which appears in Theorem 3.6 is really optimal.

Question 3.26. Prove that the sheaf $\mathcal{B} \subseteq \Omega^1_Y(\log D)$ is the smallest sheaf possible for which Theorem 3.6 holds, or else find the smallest possible sheaf. For instance, does Theorem 3.6 admit a natural improvement if we replace $\Omega^1_Y(\log D)$ by a suitable sheaf of orbifold differentials, using Campana's language of geometric orbifolds?

4. Techniques II: Reflexive differentials on singular spaces

4.1. Motivation

4.1.1. A special case of the Viehweg conjecture To motivate the results presented in this section, let $f^\circ : X^\circ \to Y^\circ$ be a smooth, projective family of canonically polarised varieties over a smooth, quasi-projective base manifold, and assume that the family f° is of maximal variation, i.e., that $\operatorname{Var}(f^\circ) = \dim Y^\circ$. As before, choose a smooth compactification $Y \supseteq Y^\circ$ such that $D := Y \setminus Y^\circ$ is a divisor with only simple normal crossings.

To prove Viehweg's conjecture, we need to show that the logarithmic Kodaira dimension of Y° is maximal, i.e., that $\kappa(Y^\circ) = \dim Y^\circ$. In particular, we need to rule out the possibility that $\kappa(Y^\circ) = 0$. As we will see in the proof of Proposition 4.1 below, a relatively elementary argument exists in cases where the Picard number of Y is one, $\rho(Y) = 1$. We refer the reader to [HL97, Sect. I.1] for the notion of semistability and for a discussion of the Harder-Narasimhan filtration used in the proof.

Proposition 4.1 (Partial answer to Viehweg's conjecture in case $\rho(Y) = 1$)**.** *In the setup described above, if we additionally assume that $\rho(Y) = 1$, then $\kappa(Y^\circ) \neq 0$.*

Proof. We argue by contradiction and assume that both $\kappa(Y^\circ) = 0$ and that $\rho(Y) = 1$. Let $\mathcal{A} \subseteq \operatorname{Sym}^m \Omega^1_Y(\log D)$ be the big invertible sheaf whose existence is guaranteed by the Viehweg-Zuo construction, Theorem 3.1. Since $\rho(Y) = 1$, the sheaf $\mathcal{A}$ is actually ample.

As a first step, observe that the log canonical bundle $K_Y + D$ must be torsion, i.e., that there exists a number $m' \in \mathbb{N}^+$ such that $\mathcal{O}_Y\big(m' \cdot (K_Y + D)\big) \cong \mathcal{O}_Y$. This follows from the assumption that $\kappa(K_Y + D) = 0$ and from the observation that on a projective manifold with $\rho = 1$, any invertible sheaf which admits a non-zero section is either trivial or ample. In particular, we obtain that the divisor $K_Y + D$ is numerically trivial.

Next, let $C \subseteq Y$ be a general complete intersection curve in the sense of Mehta-Ramanathan, cf. [HL97, Sect. II.7]. The numerical triviality of $K_Y + D$ will then imply that

$$(K_Y + D).C = c_1\big(\Omega^1_Y(\log D)\big).C = c_1\big(\operatorname{Sym}^m \Omega^1_Y(\log D)\big).C = 0.$$

On the other hand, since $\mathcal{A}$ is ample, we have that $c_1(\mathcal{A}).C > 0$. In summary, we obtain that the symmetric product sheaf $\operatorname{Sym}^m \Omega^1_Y(\log D)$ is not semistable. Since we are working in characteristic zero, this implies that the sheaf of Kähler differentials $\Omega^1_Y(\log D)$ will likewise not be semistable, and contains a destabilising subsheaf $\mathcal{B} \subseteq \Omega^1_Y(\log D)$ with $c_1(\mathcal{B}).C > 0$, cf. [HL97, Cor. 3.2.10]. Since the intersection number $c_1(\mathcal{B}).C$ is positive, the rank r of the sheaf $\mathcal{B}$ must be strictly less than $\dim Y$, and its determinant is an ample invertible subsheaf of the sheaf

of logarithmic r-forms,

$$\det \mathcal{B} \subseteq \Omega^r_Y(\log D).$$

This, however, contradicts the Bogomolov-Sommese Vanishing Theorem 2.17 and therefore ends the proof. □

4.1.2. Application of minimal model theory The assumption that $\rho(Y) = 1$ is not realistic. In the general situation, where $\rho(Y)$ can be arbitrarily large, we will apply the minimal model program to the pair (Y, D). As we will sketch in Section 5, assuming that the standard conjectures of minimal model theory hold true, a run of the minimal model program for a suitable choice of a boundary divisor will yield a diagram,

$$\begin{tikzcd} Y \arrow[r, dashed, "\lambda", "\text{minimal model program}"'] & Y_\lambda \arrow[d, "\pi"', "\text{fibre space}"] \\ & Z_\lambda, \end{tikzcd}$$

where $\lambda : Y \dashrightarrow Y_\lambda$ is a birational map whose inverse does not contract any divisors, and where either $\rho(Y_\lambda) = 1$ and Z_λ is a point, or where Y_λ has the structure of a proper Mori– or Kodaira fibre space. In the first case, we can try to copy the proof of Proposition 4.1 above. In the second case, we can use the fibre structure and try to argue inductively.

The main problem that arises when adopting the proof of Proposition 4.1 is the presence of singularities. Both the space Y_λ and the cycle-theoretic image divisor $D_\lambda \subset Y_\lambda$ will generally be singular, and the pair (Y_λ, D_λ) will generally be dlt. This leads to two difficulties.

(1) The sheaf $\Omega^1_{Y_\lambda}(\log D_\lambda)$ of logarithmic Kähler differentials is generally not pure in the sense of [HL97, Sect. 1.1]. Accordingly, there is no good notion of stability that would be suitable to construct a Harder-Narasimhan filtration.

(2) The Viehweg-Zuo construction does not work for singular varieties. The author is not aware of any method suitable to prove positivity results for Kähler differentials, or prove the existence of sections in any symmetric product of $\Omega^1_{Y_\lambda}(\log D_\lambda)$.

The aim of the present Section 4 is to show that that both problems can be overcome if we replace the sheaf $\Omega^1_{Y_\lambda}(\log D_\lambda)$ of Kähler differentials by its double dual $\Omega^{[1]}_{Y_\lambda}(\log D_\lambda) := \big(\Omega^1_{Y_\lambda}(\log D_\lambda)\big)^{**}$. We refer to [Rei87, Sect. 1.6] for a discussion of the double dual in this context, and to [OSS80, II. Sect. 1.1] for a thorough discussion of reflexive sheaves. The following notation will be useful in the discussion.

Notation 4.2 (Reflexive tensor operations)**.** *Let X be a normal variety and $\mathcal{A}$ a coherent sheaf of $\mathcal{O}_X$-modules. Given any number $n \in \mathbb{N}$, set $\mathcal{A}^{[n]} := (\mathcal{A}^{\otimes n})^{**}$,*

$\operatorname{Sym}^{[n]} \mathcal{A} := (\operatorname{Sym}^n \mathcal{A})^{**}$. *If* $\pi : X' \to X$ *is a morphism of normal varieties, set* $\pi^{[*]}(\mathcal{A}) := \big(\pi^* \mathcal{A}\big)^{**}$. *In a similar vein, set* $\Omega_X^{[p]} := \big(\Omega_X^p\big)^{**}$ *and* $\Omega_X^{[p]}(\log D) := \big(\Omega_X^p(\log D)\big)^{**}$.

Notation 4.3 (Reflexive differential forms)**.** *A section in* $\Omega_X^{[p]}$ *or* $\Omega_X^{[p]}(\log D)$ *will be called a* reflexive form *or a* reflexive logarithmic form*, respectively.*

Fact 4.4 (Torsion freeness and Harder-Narasimhan filtration). Reflexive sheaves are torsion free and therefore pure. In particular, a Harder-Narasimhan filtration exists for $\Omega_X^{[p]}(\log D)$ and for the symmetric products $\operatorname{Sym}^{[n]} \Omega_X^1(\log D)$.

Fact 4.5 (Extension over small sets). If X is a normal space, if $\mathcal{A}$ is any reflexive sheaf on X and if $Z \subset X$ any set of $\operatorname{codim}_X Z \geq 2$, then the restriction map

$$H^0\big(X,\, \mathcal{A}\big) \to H^0\big(X \setminus Z,\, \mathcal{A}\big)$$

is in fact isomorphic. We say that "sections in $\mathcal{A}$ extend over the small set Z".

If $U := X \setminus Z$ is the complement of Z, with inclusion map $\iota : U \to X$, it follows immediately that $\mathcal{A} = \iota_*(\mathcal{A}|_U)$. In a similar vein, if $\mathcal{B}_U$ is any locally free sheaf on U, its push-forward sheaf $\iota_*(\mathcal{B}_U)$ will always be reflexive.

4.1.3. Outline of this section It follows almost by definition that sheaves of reflexive differentials have very good push-forward properties. In Section 4.2 we will use these properties to overcome one of the problems mentioned above and to produce Viehweg-Zuo sheaves of reflexive differentials on singular spaces. Perhaps more importantly, we will in Section 4.3 recall extension results for log canonical varieties. These results show that reflexive differentials often admit a pull-back map, similar to the standard pull-back of Kähler differentials. A generalisation of the Bogomolov-Sommese vanishing theorem to log canonical varieties follows as a corollary.

In Section 4.4, we recall that some of the most important constructions known for logarithmic differentials on snc pairs also work for reflexive differentials on dlt pairs. This includes the existence of a residue sequence. For our purposes, this makes reflexive differentials almost as useful as regular differentials in the theory of smooth spaces. As we will roughly sketch in Section 5, these results will allow to adapt the proof of Proposition 4.1 to the singular setup, and will give a proof of Viehweg's Conjecture 2.8, at least for families over base manifolds of dimension ≤ 3. Section 4.5 gives a brief sketch of the proof of the pull-back result of Section 4.3. We end by mentioning a few open problems and conjectures.

Some of the material presented in the current Section 4, including Section 4.4 and all the illustrations, is taken without much modification from the paper [GKKP11]. Section 4.2 follows the paper [KK08c].

4.2. Existence of a push-forward map

Fact 4.5 implies that any Viehweg-Zuo sheaf which exists on a pair (Z, Δ) of a smooth variety and a reduced divisor with simple normal crossing support immediately implies the existence of a Viehweg-Zuo sheaf of reflexive differentials on any minimal model of (Z, Δ), and that the Kodaira-Iitaka dimension only increases in the process. To formulate the result precisely, we briefly recall the definition of the Kodaira-Iitaka dimension for reflexive sheaves.

Definition 4.6 (Kodaira-Iitaka dimension of a sheaf, [KK08c, Not. 2.2])**.** *Let Z be a normal projective variety and $\mathcal{A}$ a reflexive sheaf of rank one on Z. If $h^0\big(Z,\, \mathcal{A}^{[n]}\big) = 0$ for all $n \in \mathbb{N}$, then we say that $\mathcal{A}$ has Kodaira-Iitaka dimension $\kappa(\mathcal{A}) := -\infty$. Otherwise, set*

$$M := \big\{n \in \mathbb{N} \,|\, h^0\big(Z,\, \mathcal{A}^{[n]}\big) > 0\big\},$$

recall that the restriction of $\mathcal{A}$ to the smooth locus of Z is locally free and consider the natural rational mapping

$$\phi_n : Z \dashrightarrow \mathbb{P}\big(H^0\big(Z,\, \mathcal{A}^{[n]}\big)^*\big) \text{ for each } n \in M.$$

The Kodaira-Iitaka dimension of $\mathcal{A}$ is then defined as

$$\kappa(\mathcal{A}) := \max_{n \in M}\big(\dim \overline{\phi_n(Z)}\big).$$

With this notation, the main result concerning the push-forward is then formulated as follows.

Proposition 4.7 (Push forward of Viehweg-Zuo sheaves, [KK08c, Lem. 5.2])**.** *Let (Z, Δ) be a pair of a smooth variety and a reduced divisor with simple normal crossing support. Assume that there exists a reflexive sheaf $\mathcal{A} \subseteq \operatorname{Sym}^{[n]} \Omega^1_Z(\log \Delta)$ of rank one. If $\lambda : Z \dashrightarrow Z'$ is a birational map whose inverse does not contract any divisor, if Z' is normal and Δ' is the (necessarily reduced) cycle-theoretic image of Δ, then there exists a reflexive sheaf $\mathcal{A}' \subseteq \operatorname{Sym}^{[n]} \Omega^1_{Z'}(\log \Delta')$ of rank one, and of Kodaira-Iitaka dimension $\kappa(\mathcal{A}') \geq \kappa(\mathcal{A})$.*

Proof. The assumption that λ^{-1} does not contract any divisors and the normality of Z' guarantee that $\lambda^{-1} : Z' \dashrightarrow Z$ is a well-defined embedding over an open subset $U \subseteq Z'$ whose complement has codimension $\operatorname{codim}_{Z'}(Z' \setminus U) \geq 2$, cf. Zariski's main theorem [Har77, V 5.2]. In particular, $\Delta'|_U = \big(\lambda^{-1}|_U\big)^{-1}(\Delta)$. Let $\iota : U \to Z'$ denote the inclusion and set $\mathcal{A}' := \iota_*\big((\lambda^{-1}|_U)^*\mathcal{A}\big)$ —this sheaf is reflexive by Fact 4.5. We obtain an inclusion of reflexive sheaves, $\mathcal{A}' \subseteq \operatorname{Sym}^{[n]} \Omega^1_{Z'}(\log \Delta')$. By construction, we have that $h^0\big(Z',\, \mathcal{A}'^{[m]}\big) \geq h^0(Z,\, \mathcal{A}^{[m]})$ for all $m > 0$, hence $\kappa(\mathcal{A}') \geq \kappa(\mathcal{A})$. □

Given the importance of the Viehweg-Zuo construction, Theorem 3.1, we will call the sheaves $\mathcal{A}$ which appear in Proposition 4.7 "Viehweg-Zuo sheaves".

Notation 4.8 (Viehweg-Zuo sheaves). *Let (Z, Δ) be a pair of a smooth variety and a reduced divisor with simple normal crossing support, and let $n \in \mathbb{N}$ be any number. A reflexive sheaf $\mathcal{A} \subseteq \operatorname{Sym}^{[n]} \Omega^1_Z(\log \Delta)$ of rank one will be called a "Viehweg-Zuo sheaf".*

4.3. Existence of a pull-back morphism, statement and applications

Kähler differentials are characterised by a number of universal properties, one of the most important being the existence of a pull-back map: if $\gamma : Z \to X$ is any morphism of algebraic varieties and if $p \in \mathbb{N}$, then there exists a canonically defined sheaf morphism

$$d\gamma : \gamma^* \Omega^p_X \to \Omega^p_Z. \tag{4.9}$$

The following example illustrates that for sheaves of reflexive differentials on normal spaces, a pull-back map does not exist in general.

Example 4.10 (Pull-back morphism for dualizing sheaves, cf. [GKKP11, Ex. 4.2]). Let X be a normal Gorenstein variety of dimension n, and let $\gamma : Z \to X$ be any resolution of singularities. Observing that the sheaf of reflexive n-forms is precisely the dualizing sheaf, $\Omega^{[n]}_X \simeq \omega_X$, it follows directly from the definition of canonical singularities that X has canonical singularities if and only if a pull-back morphism $d\gamma : \gamma^* \Omega^{[n]}_X \to \Omega^n_Z$ exists.

Together with Daniel Greb, Sándor Kovács and Thomas Peternell, the author has shown that a pull-back map for reflexive differentials always exists if the target is log canonical.

Theorem 4.11 (Pull-back map for differentials on lc pairs, [GKKP11, Thm. 4.3]). *Let (X, D) be an log canonical pair, and let $\gamma : Z \to X$ be a morphism from a normal variety Z such that the image of Z is not contained in the reduced boundary or in the singular locus, i.e.,*

$$\gamma(Z) \not\subseteq (X, D)_{\mathrm{sing}} \cup \operatorname{supp}\lfloor D \rfloor.$$

If $1 \leq p \leq \dim X$ is any index and

$$\Delta := \textit{largest reduced Weil divisor contained in } \gamma^{-1}\big(\textit{non-klt locus}\big),$$

then there exists a sheaf morphism,

$$d\gamma : \gamma^* \Omega^{[p]}_X(\log \lfloor D \rfloor) \to \Omega^{[p]}_Z(\log \Delta),$$

that agrees with the usual pull-back morphism (4.9) of Kähler differentials at all points $p \in Z$ where $\gamma(p) \notin (X, D)_{\mathrm{sing}} \cup \operatorname{supp}\lfloor D \rfloor$.

Remark 4.12. If follows from the definition of klt, [KM98, Def. 2.34], that the components of D which appear with coefficient one are always contained in the non-klt locus of (X, D). In particular, the divisor Δ defined in Theorem 4.11 always contains the codimension-one part of $\gamma^{-1}\big(\operatorname{supp}\lfloor D \rfloor\big)$.

The assertion of Theorem 4.11 is rather general and perhaps a bit involved. For klt spaces, the statement reduces to the following simpler result.

Theorem 4.13 (Pull-back map for differentials on klt spaces)**.** *Let X be a normal klt variety*[4]*, and let $\gamma : Z \to X$ be a morphism from a normal variety Z such that the image $\gamma(Z)$ is not entirely contained in the singular locus of X. If $1 \leq p \leq \dim X$ is any index then there exists a sheaf morphism,*

$$d\gamma : \gamma^* \Omega_X^{[p]} \to \Omega_Z^{[p]},$$

that agrees on an open set with the usual pull-back morphism of Kähler differentials. □

Extension properties of differential forms that are closely related to the existence of pull-back maps have been studied in the literature, mostly considering only special values of p. Using Steenbrink's generalization of the Grauert-Riemenschneider vanishing theorem as their main input, similar results were shown by Steenbrink and van Straten for varieties X with only isolated singularities and for $p \leq \dim X - 2$, without any further assumption on the nature of the singularities, [SvS85, Thm. 1.3]. Flenner extended these results to normal varieties, subject to the condition that $p \leq \operatorname{codim} X_{\text{sing}} - 2$, [Fle88]. Namikawa proved the extension properties for $p \in \{1, 2\}$, in case X has canonical Gorenstein singularities, [Nam01, Thm. 4]. In the case of finite quotient singularities similar results were obtained in [dJS04]. For a log canonical pair with reduced boundary divisor, the cases $p \in \{1, \dim X - 1, \dim X\}$ were settled in [GKK10, Thm. 1.1].

A related setup where the pair (X, D) is snc, and where $\pi : \widetilde{X} \to X$ is the composition of a finite Galois covering and a subsequent resolution of singularities has been studied by Esnault and Viehweg. In [EV82] they obtain in their special setting similar results and additionally prove vanishing of higher direct image sheaves.

A brief sketch of the proof of Theorem 4.11 is given in Section 4.5 below. The proof uses a strengthening of the Steenbrink vanishing theorem, which follows from local Hodge-theoretic properties of log canonical singularities, in particular from the fact that log canonical spaces are Du Bois. These methods are combined with results available only for special classes of singularities, such as the recent progress in minimal model theory and partial classification of singularities that appear in minimal models.

4.3.1. Applications Theorem 4.11 has many applications useful for moduli theory. We mention two applications which will be important in our context. The first application generalises the Bogomolov-Sommese vanishing theorem to singular spaces.

[4] More precisely, we should say "Let X be a normal variety such that the pair $(X, \emptyset)$ is klt…"

Corollary 4.14 (Bogomolov-Sommese vanishing for lc pairs, [GKKP11, Thm. 7.2])**.** *Let (X, D) be a log canonical pair, where X is projective. If $\mathcal{A} \subseteq \Omega_X^{[p]}(\log\lfloor D\rfloor)$ is a $\mathbb{Q}$-Cartier reflexive subsheaf of rank one, then $\kappa(\mathcal{A}) \leq p$.*

Remark 4.15 (Notation used in Corollary 4.14)*.* The number $\kappa(\mathcal{A})$ appearing in the statement of Corollary 4.14 is the generalised Kodaira-Iitaka dimension introduced in Definition 4.6. A reflexive sheaf $\mathcal{A}$ is rank one is called $\mathbb{Q}$-Cartier if there exists a number $n \in \mathbb{N}^+$ such that the nth reflexive tensor product $\mathcal{A}^{[n]}$ is invertible.

Proof of Corollary 4.14 in a special case. We prove Corollary 4.14 only in the special case where the sheaf $\mathcal{A} \subseteq \Omega_X^{[p]}(\log\lfloor D\rfloor)$ is invertible. The reader interested in a full proof is referred to the original reference [GKKP11].

Let $\gamma : Z \to X$ be any resolution of singularities, and let $\Delta \subset Z$ be the reduced divisor defined in Theorem 4.11 above. Theorem 4.11 will then assert the existence of an inclusion

$$\gamma^*(\mathcal{A}) \to \Omega_Z^p(\log\Delta),$$

and the standard Bogomolov-Sommese vanishing result, Theorem 2.17, applies to give that $\kappa\big(\gamma^*(\mathcal{A})\big) \leq p$. Since $\mathcal{A}$ is invertible, and since γ is birational, it is clear that $\kappa\big(\gamma^*(\mathcal{A})\big) = \kappa(\mathcal{A})$, finishing the proof. □

Warning 4.16. Taking the double dual of a sheaf does generally *not* commute with pulling back. Since reflexive tensor products were used in Definition 4.6 to define the Kodaira-Iitaka dimension of a sheaf, it is generally false that the Kodaira-Iitaka dimension stays invariant when pulling a sheaf $\mathcal{A}$ back to a resolution of singularities. The proof of Corollary 4.14 which is given in the simple case where $\mathcal{A}$ is invertible does therefore not work without substantial modification in the general setup where $\mathcal{A}$ is only $\mathbb{Q}$-Cartier.

The second application of Theorem 4.11 concerns rationally chain connected singular spaces. Rationally chain connected manifolds are rationally connected, and do not carry differential forms. Building on work of Hacon and McKernan, [HM07], we show that the same holds for reflexive forms on klt pairs.

Corollary 4.17 (Reflexive differentials on rationally chain connected spaces, [GKKP11, Thm. 5.1])**.** *Let (X, D) be a klt pair. If X is rationally chain connected, then X is rationally connected, and $H^0\big(X, \Omega_X^{[p]}\big) = 0$ for all $p \in \mathbb{N}, 1 \leq p \leq \dim X$.*

Proof. Choose a log resolution of singularities, $\pi : \widetilde{X} \to X$ of the pair (X, D). Since klt pairs are also dlt, a theorem of Hacon-McKernan, [HM07, Cor. 1.5(2)], applies to show that X and $\widetilde{X}$ are both rationally connected. In particular, it follows that $H^0\big(\widetilde{X}, \Omega^p_{\widetilde{X}}\big) = 0$ for all $p > 0$ by [Kol96, IV. Cor. 3.8].

Since (X, D) is klt, Theorem 4.11 asserts that there exists a pull-back morphism $d\pi : \pi^*\Omega_X^{[p]} \to \Omega^p_{\widetilde{X}}$. As π is birational, $d\pi$ is generically injective and since

$\Omega_X^{[p]}$ is torsion-free, this means that the induced morphism on the level of sections is injective:

$$\pi^* : H^0\big(X,\, \Omega_X^{[p]}\big) \to H^0\big(\widetilde{X},\, \Omega^p_{\widetilde{X}}\big) = 0.$$

The claim then follows. □

4.4. Residue theory and restrictions for differentials on dlt pairs

Logarithmic Kähler differentials on snc pairs are canonically defined. They are characterised by strong universal properties and appear accordingly in a number of important sequences, filtered complexes and other constructions. First examples include the following:

(1) the pull-back property of differentials under arbitrary morphisms,
(2) relative differential sequences for smooth morphisms,
(3) residue sequences associated with snc pairs, and
(4) the description of Chern classes as the extension classes of the first residue sequence.

Reflexive differentials do in general not enjoy the same universal properties as Kähler differentials. However, we have seen in Theorem 4.11 that reflexive differentials do have good pull-back properties if we are working with log canonical pairs. In the present Section 4.4, we would like to make the point that each of the other properties listed above also has a very good analogue for reflexive differentials, as long as we are working with dlt pairs. This makes reflexive differential extremely useful in practise. In a sense, it seems fair to say that "reflexive differentials and dlt pairs are made for one another".

4.4.1. The relative differential sequence for snc pairs Here we recall the generalisation of the standard sequence for relative differentials, [Har77, Prop. II.8.11], to the logarithmic setup. For this, we introduce the notion of an *snc morphism* as a logarithmic analogue of a smooth morphism. Although *relatively snc divisors* have long been used in the literature, cf. [Del70, Sect. 3], we are not aware of a good reference that discusses them in detail. Recall that a pair (X, D) is called an "snc pair" if X is smooth, and if the divisor D is reduced and has only simple normal crossing support.

Notation 4.18 (Intersection of boundary components)**.** *Let (X, D) be a pair of a normal space X and a divisor D, where D is written as a sum of its irreducible components $D = \alpha_1 D_1 + \ldots + \alpha_n D_n$. If $I \subseteq \{1, \ldots, n\}$ is any non-empty subset, we consider the scheme-theoretic intersection $D_I := \cap_{i\in I} D_i$. If I is empty, set $D_I := X$.*

Remark 4.19 (Description of snc pairs). In the setup of Notation 4.18, it is clear that the pair (X, D) is snc if and only if all D_I are smooth and of codimension equal to the number of defining equations: $\operatorname{codim}_X D_I = |I|$ for all I where $D_I \neq \emptyset$.

Definition 4.20 (Snc morphism, relatively snc divisor, [VZ02, Def. 2.1])**.** *If (X, D) is an snc pair and $\phi : X \to T$ a surjective morphism to a smooth variety, we say that D is* relatively snc*, or that ϕ is* an snc morphism of the pair (X, D) *if for any set I with $D_I \neq \emptyset$ all restricted morphisms $\phi|_{D_I} : D_I \to T$ are smooth of relative dimension* $\dim X - \dim T - |I|$.

Remark 4.21 (Fibers of an snc morphisms)*.* If (X, D) is an snc pair and $\phi : X \to T$ is any surjective snc morphism of (X, D), it is clear from Remark 4.19 that if $t \in T$ is any point, with preimages $X_t := \phi^{-1}(t)$ and $D_t := D \cap X_t$ then the pair (X_t, D_t) is again snc.

Remark 4.22 (All morphisms are generically snc)*.* If (X, D) is an snc pair and $\phi : X \to T$ is any surjective morphism, it is clear from generic smoothness that there exists a dense open set $T^\circ \subseteq T$, such that $D \cap \phi^{-1}(T^\circ)$ is relatively snc over T°.

Let (X, D) be a reduced snc pair, and $\phi : X \to T$ an snc morphism of (X, D), as introduced in Definition 4.20. In this setting, the standard pull-back morphism of 1-forms extends to yield the following exact sequence of locally free sheaves on X,

$$0 \to \phi^*\Omega^1_T \to \Omega^1_X(\log D) \to \Omega^1_{X/T}(\log D) \to 0, \tag{4.23}$$

called the "relative differential sequence for logarithmic differentials". We refer to [EV90, Sect. 4.1] [Del70, Sect. 3.3] or [BDIP02, p. 137ff] for a more detailed explanation. For forms of higher degrees, the sequence (4.23) induces filtration

$$\Omega^p_X(\log D) = \mathcal{F}^0(\log) \supseteq \mathcal{F}^1(\log) \supseteq \cdots \supseteq \mathcal{F}^p(\log) \supseteq \{0\} \tag{4.24}$$

with quotients

$$0 \to \mathcal{F}^{r+1}(\log) \to \mathcal{F}^r(\log) \to \phi^*\Omega^r_T \otimes \Omega^{p-r}_{X/T}(\log D) \to 0 \tag{4.25}$$

for all r. We refer to [Har77, Ex. II.5.16] for the construction of (4.24).

The main result of this section, Theorem 4.26, gives analogues of (4.23)–(4.25) in case where (X, D) is dlt. In essence, Theorem 4.26 says that all properties of the relative differential sequence still hold on dlt pairs if one removes from X a set Z of codimension $\operatorname{codim}_X Z \geq 3$.

Theorem 4.26 (Relative differential sequence on dlt pairs, [GKKP11, Thm. 10.6])**.** *Let (X, D) be a dlt pair with X connected. Let $\phi : X \to T$ be a surjective morphism to a normal variety T. Then, there exists a non-empty smooth open subset $T^\circ \subseteq T$ with preimages $X^\circ = \phi^{-1}(T^\circ)$, $D^\circ = D \cap X^\circ$, and a filtration*

$$\Omega^{[p]}_{X^\circ}(\log\lfloor D^\circ\rfloor) = \mathcal{F}^{[0]}(\log) \supseteq \cdots \supseteq \mathcal{F}^{[p]}(\log) \supseteq \{0\} \tag{4.27}$$

on X° with the following properties.

(1) *The filtrations* (4.24) *and* (4.27) *agree wherever the pair* $(X^\circ, \lfloor D^\circ \rfloor)$ *is snc, and* ϕ *is an snc morphism of* $(X^\circ, \lfloor D^\circ \rfloor)$*.*
(2) *For any* r*, the sheaf* $\mathcal{F}^{[r]}(\log)$ *is reflexive, and* $\mathcal{F}^{[r+1]}(\log)$ *is a saturated subsheaf of* $\mathcal{F}^{[r]}(\log)$*.*
(3) *For any* r*, there exists a sequence of sheaves of* $\mathcal{O}_{X^\circ}$*-modules,*

$$0 \to \mathcal{F}^{[r+1]}(\log) \to \mathcal{F}^{[r]}(\log) \to \phi^*\Omega^r_{T^\circ} \otimes \Omega^{[p-r]}_{X^\circ/T^\circ}(\log\lfloor D^\circ \rfloor) \to 0,$$

which is exact and analytically locally split in codimension 2*.*
(4) *There exists an isomorphism* $\mathcal{F}^{[p]}(\log) \simeq \phi^*\Omega^p_{T^\circ}$*.*

Remark 4.28 (Notation used in Theorem 4.26). If S is any complex variety, we call a sequence of sheaf morphisms,

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0, \tag{4.29}$$

"exact and analytically locally split in codimension 2" if there exists a closed subvariety $C \subset S$ of codimension $\operatorname{codim}_S C \geq 3$ and a covering of $S \setminus C$ by subsets $(U_i)_{i\in I}$ which are open in the analytic topology, such that the restriction of (4.29) to $S \setminus C$ is exact, and such that the restriction of (4.29) to any of the open sets U_i splits. We refer to Footnote 2 on Page 9 for references concerning the notion of a "dlt pair".

Idea of proof of Theorem 4.26. We give only a very rough and incomplete idea of the proof of Theorem 4.26. To construct the filtration in (4.27), one takes the filtration (4.24) which exists on the open set $X \setminus X_{\text{sing}}$ wherever the morphism ϕ is snc, and extends the sheaves to reflexive sheaves that are defined on all of X. It is then not very difficult to show that the sequences (4.26.3) are exact and locally split away from a subset $Z \subset X$ of codimension $\operatorname{codim}_X Z \geq 2$. The main point of Theorem 4.26 is, however, that it suffices to remove from X a set of codimension $\operatorname{codim}_X Z \geq 3$. For this, a careful analysis of the codimension-two structure of dlt pairs, cf. [GKKP11, Sect. 9], proves to be key. □

4.4.2. Residue sequences for reflexive differential forms A very important feature of logarithmic differentials is the existence of a residue map. In its simplest form consider a smooth hypersurface $D \subset X$ in a manifold X. The residue map is then the cokernel map in the exact sequence

$$0 \to \Omega^1_X \to \Omega^1_X(\log D) \to \mathcal{O}_D \to 0.$$

More generally, consider a reduced snc pair (X, D). Let $D_0 \subseteq D$ be any irreducible component and recall from [EV92, 2.3(b)] that there exists a residue sequence,

$$0 \to \Omega^p_X(\log(D - D_0)) \longrightarrow \Omega^p_X(\log D) \xrightarrow{\rho^p} \Omega^{p-1}_{D_0}(\log D_0^c) \to 0,$$

where $D_0^c := (D - D_0)|_{D_0}$ denotes the "restricted complement" of D_0. More generally, if $\phi : X \to T$ is an snc morphism of (X, D) we have a relative residue

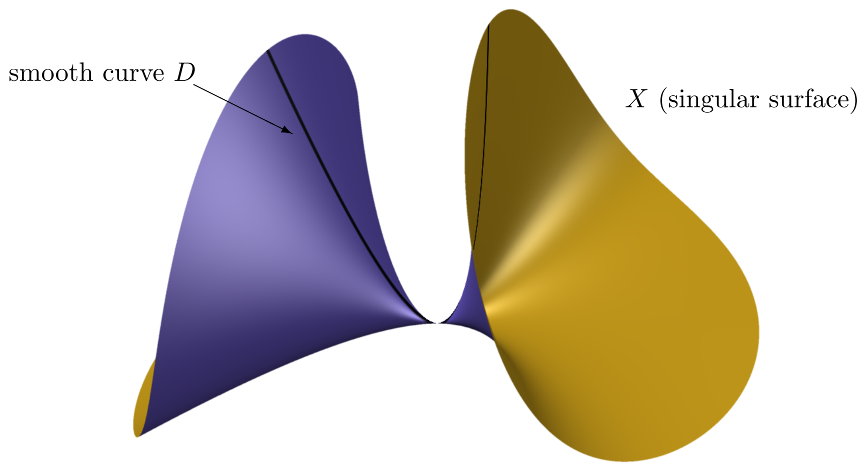


FIGURE 1. A setup for the residue map on singular spaces.

sequence

$$(4.30)\quad 0 \to \Omega^p_{X/T}(\log(D-D_0)) \longrightarrow \Omega^p_{X/T}(\log D) \xrightarrow{\rho^p} \Omega^{p-1}_{D_0/T}(\log D_0^c) \to 0.$$

The sequence (4.30) is not a sequence of locally free sheaves on X, and its restriction to D_0 will never be exact on the left. However, an elementary argument, cf. [KK08a, Lem. 2.13.2], shows that restriction of (4.30) to D_0 induces the following exact sequence

$$(4.31)\quad 0 \to \Omega^p_{D_0/T}(\log D_0^c) \xrightarrow{i^p} \Omega^p_{X/T}(\log D)|_{D_0} \xrightarrow{\rho^p_D} \Omega^{p-1}_{D_0/T}(\log D_0^c) \to 0,$$

which is very useful for inductive purposes. We recall without proof the following elementary fact about the residue sequence.

Fact 4.32 (Residue map as a test for logarithmic poles). If $\sigma \in H^0\big(X,\, \Omega^p_{X/T}(\log D)\big)$ is any reflexive form, then $\sigma \in H^0\big(X,\, \Omega^p_{X/T}(\log(D-D_0))\big)$ if and only if $\rho^p(\sigma) = 0$.

If the pair (X, D) is not snc, no residue map exists in general. However, if (X, D) is dlt, then [KM98, Cor. 5.52] applies to show that D_0 is normal, and an analogue of the residue map ρ^p exists for sheaves of reflexive differentials. To illustrate the problem we are dealing with, consider a normal space X that contains a smooth Weil divisor $D = D_0$, similar to the one sketched in Figure 1. One can easily construct examples where the singular set $Z := X_{\mathrm{sing}}$ is contained in D and has codimension 2 in X, but codimension one in D. In this setting, a reflexive form $\sigma \in H^0\big(D_0,\, \Omega^{[p]}_X(\log D_0)|_{D_0}\big)$ is simply the restriction of a logarithmic form defined outside of Z, and the form $\rho^{[p]}(\sigma)$ is the extension of the well-defined form $\rho^p(\sigma|_{D_0\setminus Z})$ over Z, as a rational form with poles along $Z \subset D_0$. If the

singularities of X are bad, it will generally happen that the extension $\rho^{[p]}(\sigma)$ has poles of arbitrarily high order. Theorem 4.33 asserts that this does not happen when (X, D) is dlt.

Theorem 4.33 (Residue sequences for dlt pairs, [GKKP11, Thm. 11.7])**.** *Let (X, D) be a dlt pair with $\lfloor D \rfloor \neq \emptyset$ and let $D_0 \subseteq \lfloor D \rfloor$ be an irreducible component. Let $\phi : X \to T$ be a surjective morphism to a normal variety T such that the restricted map $\phi|_{D_0} : D_0 \to T$ is still surjective. Then, there exists a non-empty open subset $T^\circ \subseteq T$, such that the following holds if we denote the preimages as $X^\circ = \phi^{-1}(T^\circ)$, $D^\circ = D \cap X^\circ$, and the "complement" of D_0° as $D_0^{\circ,c} := \big(\lfloor D^\circ \rfloor - D_0^\circ\big)|_{D_0^\circ}$.*

(1) *There exists a sequence*

$$0 \to \Omega^{[r]}_{X^\circ/T^\circ}(\log(\lfloor D^\circ \rfloor - D_0^\circ)) \to \Omega^{[r]}_{X^\circ/T^\circ}(\log\lfloor D^\circ \rfloor) \xrightarrow{\rho^{[r]}} \Omega^{[r-1]}_{D_0^\circ/T^\circ}(\log D_0^{\circ,c}) \to 0$$

which is exact in X° outside a set of codimension at least 3. This sequence coincides with the usual residue sequence (4.30) wherever the pair (X°, D°) is snc and the map $\phi^\circ : X^\circ \to T^\circ$ is an snc morphism of (X°, D°).

(2) *The restriction of Sequence (4.33.1) to D_0 induces a sequence*

$$0 \to \Omega^{[r]}_{D_0^\circ/T^\circ}(\log D_0^{\circ,c}) \to \Omega^{[r]}_{X^\circ/T^\circ}(\log\lfloor D^\circ \rfloor)|^{**}_{D_0^\circ} \xrightarrow{\rho^{[r]}_{D_0^\circ}} \Omega^{[r-1]}_{D_0^\circ/T^\circ}(\log D_0^{\circ,c}) \to 0$$

which is exact on D_0° outside a set of codimension at least 2 and coincides with the usual restricted residue sequence (4.31) wherever the pair (X°, D°) is snc and the map $\phi^\circ : X^\circ \to T^\circ$ is an snc morphism of (X°, D°). □

As before, the proof of Theorem 4.33 relies on our knowledge of the codimension-two structure of dlt pairs. Fact 4.32 and Theorem 4.33 together immediately imply that the residue map for reflexive differentials can be used to check if a reflexive form has logarithmic poles along a given boundary divisor.

Remark 4.34 (Residue map as a test for logarithmic poles). In the setting of Theorem 4.33, if $\sigma \in H^0\big(X, \Omega^{[p]}_X(\log\lfloor D \rfloor)\big)$ is any reflexive form, then $\sigma \in H^0\big(X, \Omega^{[p]}_X(\log\lfloor D \rfloor - D_0)\big)$ if and only if $\rho^{[p]}(\sigma) = 0$.

4.4.3. The residue map for 1-forms Let X be a smooth variety and $D \subset X$ a smooth, irreducible divisor. The first residue sequence (4.30) of the pair (X, D) then reads

$$0 \to \Omega^1_D \to \Omega^1_X(\log D)|_D \xrightarrow{\rho^1} \mathcal{O}_D \to 0,$$

and we obtain a connecting morphism of the long exact cohomology sequence,

$$\delta : H^0\big(D, \mathcal{O}_D\big) \to H^1\big(D, \Omega^1_D\big).$$

In this setting, the standard description of the first Chern class in terms of the connecting morphism, [Har77, III. Ex. 7.4], asserts that

$$c_1\big(\mathscr{O}_X(D)|_D\big) = \delta(\mathbf{1}_D) \in H^1\big(D, \Omega^1_D\big), \tag{4.35}$$

where $\mathbf{1}_D$ is the constant function on D with value one. Theorem 4.36 generalises Identity (4.35) to the case where (X, D) is a reduced dlt pair with irreducible boundary divisor.

Theorem 4.36 (Description of Chern classes, [GKKP11, Thm. 12.2])**.** *Let (X, D) be a dlt pair, $D = \lfloor D \rfloor$ irreducible. Then, there exists a closed subset $Z \subset X$ with $\operatorname{codim}_X Z \geq 3$ and a number $m \in \mathbb{N}$ such that mD is Cartier on $X^\circ := X \setminus Z$, such that $D^\circ := D \cap X^\circ$ is smooth, and such that the restricted residue sequence*

$$0 \to \Omega^1_D \to \Omega^{[1]}_X(\log D)|^{**}_D \xrightarrow{\rho_D} \mathscr{O}_D \to 0 \tag{4.37}$$

defined in Theorem 4.33 is exact on D°. Moreover, for the connecting homomorphism δ in the associated long exact cohomology sequence

$$\delta : H^0\big(D^\circ,\, \mathscr{O}_{D^\circ}\big) \to H^1\big(D^\circ,\, \Omega^1_{D^\circ}\big)$$

we have

$$\delta(m \cdot \mathbf{1}_{D^\circ}) = c_1\left(\mathscr{O}_{X^\circ}(mD^\circ)|_{D^\circ}\right). \tag{4.38}$$

4.5. Existence of a pull-back morphism, idea of proof

The proof of Theorem 4.11 is rather involved. To illustrate the idea of the proof, we concentrate on a very special case, and give only indications what needs to be done to handle the general setup.

4.5.1. Simplifying assumptions and setup of notation The following simplifying assumptions will be maintained throughout the present Section 4.5.

Simplifying Assumptions 4.39. The space X has dimension $n := \dim X \geq 3$. It is klt, has only one single isolated singularity $x \in X$, and the divisor D is empty. The morphism $\gamma : Z \to X$ is a resolution of singularities, whose exceptional set $E \subset Z$ is a divisor with simple normal crossing support.

To prove Theorem 4.11, we need to show in essence that reflexive differential forms $\sigma \in H^0\big(X,\, \Omega^{[p]}_X\big)$ pull back to give differential forms $\widetilde{\sigma} \in H^0\big(Z,\, \Omega^p_Z\big)$. The following observation, an immediate consequence of Fact 4.5, turns out to be key.

Observation 4.40. To give a reflexive differential $\sigma \in H^0\big(X,\, \Omega^{[p]}_X\big)$, it is equivalent to give a differential form $\sigma^\circ \in H^0\big(X \setminus X_{\mathrm{sing}},\, \Omega^p_X\big)$, defined on the smooth locus of X. Since the resolution map identifies the open subvarieties $Z \setminus E$ and $X \setminus X_{\mathrm{sing}}$, we see that to give a reflexive differential $\sigma \in H^0\big(X,\, \Omega^{[p]}_X\big)$, it is in fact equivalent to give a differential form $\widetilde{\sigma}^\circ \in H^0\big(Z \setminus E,\, \Omega^p_Z\big)$.

In essence, Observation 4.40 says that to show Theorem 4.11, we need to prove that the natural restriction map

$$H^0\big(Z,\, \Omega^p_Z\big) \to H^0\big(Z \setminus E,\, \Omega^p_Z\big) \tag{4.41}$$

is in fact surjective. In other words, we need to show that any differential form on Z, which is defined outside of the γ-exceptional set E, automatically extends across E, to give a differential form defined on all of Z. This is done in two steps. We first show that the restriction map

$$H^0\big(Z,\, \Omega^p_Z(\log E)\big) \to H^0\big(Z \setminus E,\, \Omega^p_Z(\log E)\big) = H^0\big(Z \setminus E,\, \Omega^p_Z\big) \tag{4.42}$$

is surjective. In other words, we show that any differential form on Z, defined outside of E, extends as a form with logarithmic poles along E. Secondly, we show that the natural inclusion map

$$H^0\big(Z,\, \Omega^p_Z\big) \to H^0\big(Z,\, \Omega^p_Z(\log E)\big) \tag{4.43}$$

is likewise surjective. In other words, we show that globally defined differentials forms on Z, which are allowed to have logarithmic poles along E, really do not have any poles. Surjectivity of the morphisms (4.42) and (4.43) together will then imply surjectivity of (4.41), finishing the proof of Theorem 4.11.

The arguments used to prove surjectivity of (4.42) and (4.43), respectively, are of rather different nature. We will sketch the arguments in Sections 4.5.2 and 4.5.3 below.

4.5.2. Surjectivity of the restriction map (4.42) Under the Simplifying Assumptions 4.39, surjectivity of the map (4.42) has essentially been shown by Steenbrink and van Straten, [SvS85]. We give a brief synopsis of their line of argumentation and indicate additional steps of argumentation required to handle the general setting. To start, recall from [Har77, III ex. 2.3e] that the map (4.42) is part of the standard sequence that defines cohomology with supports,

$$\begin{multline} \cdots \to H^0\big(Z,\, \Omega^p_Z(\log E)\big) \to H^0\big(Z \setminus E,\, \Omega^p_Z(\log E)\big) \to \\ \to H^1_E\big(Z,\, \Omega^p_Z(\log E)\big) \to \cdots \end{multline} \tag{4.44}$$

We aim to show that the last term in (4.44) vanishes. There are two main ingredients to the proof: formal duality and Steenbrink's vanishing theorem.

Theorem 4.45 (Formal duality theorem for cohomology with support, [Har70, Chapt. 3, Thm. 3.3])**.** *Under the Assumptions 4.39, if $\mathcal{F}$ is any locally free sheaf on Z and $0 \le j \le \dim Z$ any number, then there exists an isomorphism*

$$\big((R^j\gamma_*\mathcal{F})_x\big)^\frown \cong H^{n-j}_E\big(Z,\, \mathcal{F}^* \otimes \omega_Z\big)^*,$$

where $\frown$ denotes completion with respect to the maximal ideal $\mathfrak{m}_x$ of the point $x \in X$, and where $n = \dim X = \dim Z$. ☐

A brief introduction to formal duality, together with a readable, self-contained proof of Theorem 4.45 is found in [GKK10, Appendix A] while Hartshorne's lecture notes [Har70] are the standard reference for these matters.

Theorem 4.46 (Steenbrink vanishing, [Ste85, Thm. 2.b])**.** *If p, q are any two numbers with $p+q > \dim Z$, then $R^q\gamma_*\big(\mathcal{J}_E \otimes \Omega^p_Z(\log E)\big) = 0$.* □

Remark 4.47. Steenbrink's vanishing theorem is proven using local Hodge theory of isolated singularities. For $p = n$, the sheaves Ω^n_Z and $\mathcal{J}_E \otimes \Omega^n_Z(\log E)$ are isomorphic. In this case, the Steenbrink vanishing theorem reduces to Grauert-Riemenschneider vanishing, [GR70].

Setting $\mathcal{F} := \mathcal{J}_E \otimes \Omega^{n-p}_Z(\log E)$ and using that $\mathcal{F}^* \otimes \omega_Z \cong \Omega^p_Z(\log E)$, formal duality and Steenbrink vanishing together show that $H^1_E\big(Z,\, \Omega^p_Z(\log E)\big) = 0$, for $p < \dim Z - 1$, proving surjectivity of (4.42) for these values of p. The other cases need to be treated separately.

case $\boldsymbol{p = n}$: After passing to an index-one cover, surjectivity of (4.42) in case $p = n$ follows almost directly from the definition of klt, cf. [GKK10, Sect. 5].

case $\boldsymbol{p = n-1}$: In this case one uses the duality between Ω^{n-1}_Z and the tangent sheaf T_Z, and the fact that any section in the tangent sheaf of X always lifts to the canonical resolution of singularities, cf. [GKK10, Sect. 6].

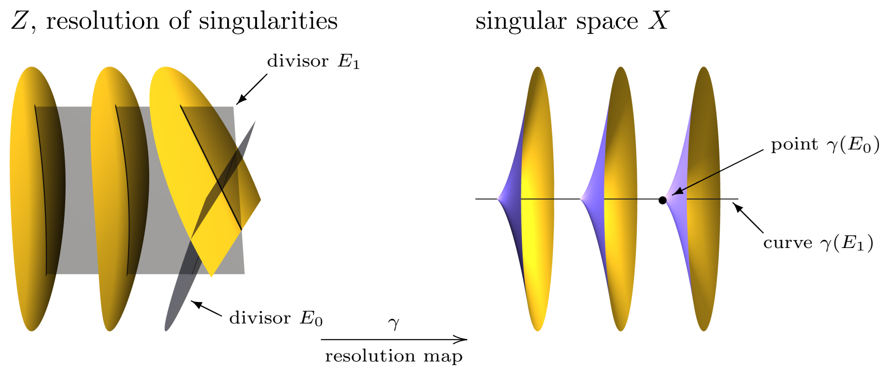
General case The argument outlined above, using formal duality and Steenbrink vanishing, works only because we were assuming that the singularities of X are isolated. In the general case, where the Simplifying Assumptions 4.39 do not necessarily hold, this is not necessarily the case. In order to deal with non-isolated singularities, one applies a somewhat involved cutting-down procedure, as indicated in Figure 2. This way, it is often possible to view non-isolated log canonical singularities a family of isolated singularities, where surjectivity of (4.42) can be shown on each member of the family. To conclude that it holds on all of Z, the following strengthening of Steenbrink vanishing is required.

Theorem 4.48 (Steenbrink-type vanishing for log canonical pairs, [GKKP11, Thm. 14.1])**.** *Let (X, D) be a log canonical pair of dimension $n \geq 2$. If $\gamma : Z \to X$ is a log resolution of singularities with exceptional set E and*

$$\Delta := \operatorname{supp}\big(E + \gamma^{-1}\lfloor D \rfloor\big),$$

then $R^{n-1}\gamma_*\big(\Omega^p_Z(\log \Delta) \otimes \mathcal{O}_Z(-\Delta)\big) = 0$ for all $0 \leq p \leq n$. □

The proof of Theorem 4.48 essentially relies on the fact that log canonical pairs are Du Bois, [KK10]. The Du Bois property generalises the notion of rational singularities. For an overview, see [KS09].



The figure sketches a situation where X is a threefold whose singular locus is a curve. Near the general point of the singular locus, the variety X looks like a family of isolated surfaces singularities. The exceptional set E of the resolution map γ contains two irreducible divisors E_0 and E_1.

FIGURE 2. Non-isolated singularities

4.5.3. Surjectivity of the inclusion map (4.43) Let $\sigma \in H^0\bigl(Z, \Omega^p_Z(\log E)\bigr)$ be any differential form on Z that is allowed to have logarithmic poles along E. To show surjectivity of the inclusion map (4.43), we need to show that σ really does not have any poles along E. To give an idea of the methods used to prove this, we consider only the case where $p > 1$. We discuss two particularly simple cases first.

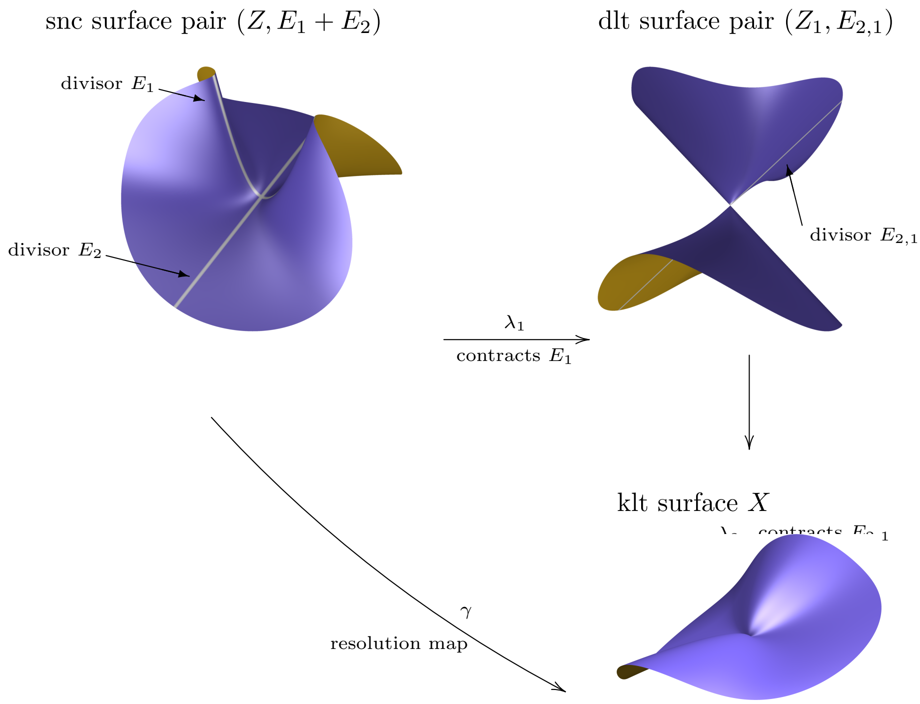
The case where E is irreducible Assume that E is irreducible. To show that σ does not have any logarithmic poles along E, recall from Fact 4.32 that it suffices to show that σ is in the kernel of the residue map

$$\rho^p : H^0\bigl(Z, \Omega^p_Z(\log E)\bigr) \to H^0\bigl(E, \Omega^{p-1}_E\bigr).$$

On the other hand, we know from a result of Hacon-McKernan, [HM07, Cor. 1.5(2)], that E is rationally connected, so that $H^0\bigl(E, \Omega^{p-1}_E\bigr) = 0$. This clearly shows that σ is in the kernel of ρ^p and completes the proof when E is irreducible.

The case where (Z, E) admits a simple minimal model program In general, the divisor E need not be irreducible. Let us therefore consider the next difficult case where E is reducible with two components, say $E = E_1 \cup E_2$. The resolution map γ will then factor via a γ-relative minimal model program of the pair (Z, E), which we assume for simplicity to have the following particularly special form, sketched[5] also in Figure 3.

[5]The computer code used to generate the images in Figure 3 is partially taken from [Bau07].



This sketch shows a resolution of an isolated klt surface singularity, and the decomposition of the resolution map given by the minimal model program of the snc pair $(Z, E_1 + E_2)$.

FIGURE 3. Resolution of an isolated klt surface singularity

$$Z = Z_0 \xrightarrow[\text{contracts } E_1 \text{ to a point}]{\lambda_1} Z_1 \xrightarrow[\text{contracts } E_{2,1} := (\lambda_1)_*(E_2) \text{ to a point}]{\lambda_2} X.$$

In this setting, the arguments outlined above apply verbatim to show that σ has no poles along the divisor E_1. To show that σ does not have any poles along the remaining component E_2, observe that it suffices to consider the induced reflexive form on the possibly singular space Z_1, say $\sigma_1 \in H^0\big(Z_1,\, \Omega^{[p]}_{Z_1}(\log E_{2,1})\big)$, where $E_{2,1} := (\lambda_1)_*(E_2)$, and to show that σ_1 does not have any poles along $E_{2,1}$. For that, we follow the same line of argument once more, accounting for the singularities of the pair $(X_1, E_{2,1})$.

The pair $(X_1, E_{2,1})$ is dlt, and it follows from adjunction that the divisor $E_{2,1}$ is necessarily normal, [KM98, Cor. 5.52]. Using the residue map for reflexive differentials on dlt pairs that was constructed in Theorem 4.33,

$$\rho^{[p]} : H^0\big(X_1,\, \Omega^{[p]}_{Z_1}(\log E_{2,1})\big) \to H^0\big(E_{2,1},\, \Omega^{[p-1]}_{E_{2,1}}\big),$$

we have seen in Remark 4.34 that it suffices to show that $\rho^{[p]}(\sigma_1) = 0$. Because the morphism λ_2 contracts the divisor $E_{2,1}$ to a point, the result of Hacon-McKernan will again apply to show that $E_{2,1}$ is rationally connected. Even though there are numerous examples of rationally connected spaces that carry non-trivial reflexive forms, we claim that in our special setup we do have the vanishing

$$H^0\big(E_{2,1},\, \Omega^{[p-1]}_{E_{2,1}}\big) = 0. \tag{4.49}$$

For this, recall from the adjunction theory for Weil divisors on normal spaces, [Kol92, Chapt. 16 and Prop. 16.5] and [Cor07, Sect. 3.9 and Glossary], that there exists a Weil divisor D_E on the normal variety $E_{2,1}$ which makes the pair $(E_{2,1}, D_E)$ klt. Now, if we knew that the extension theorem would hold for the pair $(E_{2,1}, D_E)$, we can prove the vanishing statement (4.49), arguing exactly as in the proof of Corollary 4.17, where we show the non-existence of reflexive forms on rationally connected klt spaces as a corollary of the Pull-Back Theorem 4.11. Since $\dim E_{2,1} < \dim X$, this suggests an inductive proof, beginning with easy-to-prove extension theorems for reflexive forms on surfaces, and working our way up to higher-dimensional varieties. The proof in [GKKP11] follows this inductive pattern.

The general case To handle the general case, where the Simplifying Assumptions 4.39 do not necessarily hold true, we need to work with pairs (X, D) where D is not necessarily empty, the γ-relative minimal model program might involve flips, and the singularities of X need not be isolated. All this leads to a slightly protracted inductive argument, heavily relying on cutting-down methods and outlined in detail in [GKKP11, Sect. 19].

4.6. Open problems

In view of the Viehweg-Zuo construction, it would be very interesting to know if a variant of the Pull-Back Theorem 4.11 holds for symmetric powers of $\Omega^{[1]}_X(\log D)$, or for other tensor powers. As shown by examples, cf. [GKK10, Ex. 3.1.3], the naïve generalisation of Theorem 4.11 is wrong. Still, it seems conceivable that a suitable generalisation, perhaps formulated in terms of Campana's orbifold differentials, might hold. However, note that several of the key ingredients used in the proof of Theorem 4.11, including Steenbrink's vanishing theorem, rely on (local) Hodge theory, for which no version is known for tensor powers of differential forms.

Question 4.50. Is there a formulation of the Pull-Back Theorem 4.11 that holds for symmetric and other tensor powers of differential forms?

Examples suggest that the Pull-Back Theorem 4.11 is optimal, and that the class of log canonical pairs is the natural class of spaces where a pull-back theorem can hold.

Question 4.51. To what extend is the Pull-Back Theorem 4.11 optimal? Is there a version of the pull-back theorem that does not require the log canonical divisor $K_X + D$ to be $\mathbb{Q}$-Cartier? If we are interested only in special values of p, is the divisor Δ the smallest possible?

The last question concerns the generalisation of the Bogomolov-Sommese vanishing theorem. One of the main difficulties with its current formulation is the requirement that the sheaf $\mathcal{A}$ be $\mathbb{Q}$-Cartier. We have seen in Section 4.1 how interesting reflexive subsheaves $\mathcal{A} \subseteq \Omega_X^{[p]}$ can often be constructed using the Harder-Narasimhan filtration. Unless the space X is $\mathbb{Q}$-factorial, there is, however, no way to guarantee that a sheaf constructed this way will actually be $\mathbb{Q}$-Cartier. The property to be $\mathbb{Q}$-factorial, however, is not stable under taking hyperplane sections and difficult to guarantee in practise.

Question 4.52. Is there a version of the generalised Bogomolov-Sommese vanishing theorem, Corollary 4.14, that does not require the sheaf $\mathcal{A}$ to be $\mathbb{Q}$-Cartier?

5. Viehweg's conjecture for families over threefolds, sketch of proof

5.1. A special case of the Viehweg conjecture

We conclude this paper by sketching a proof of the Viehweg Conjecture 2.8 in one special case, illustrating the use of the methods introduced in Sections 3 and 4. As in Section 4.1 we consider a family $f^\circ : X^\circ \to Y^\circ$ of canonically polarised varieties over a quasi-projective threefold. Assuming that f° is of maximal variation, we would like to show that the logarithmic Kodaira dimension $\kappa(Y^\circ)$ cannot be zero.

Proposition 5.1 (Partial answer to Viehweg's conjecture)**.** *Let $f^\circ : X^\circ \to Y^\circ$ be a smooth, projective family of canonically polarised varieties over a smooth, quasi-projective base manifold of dimension $\dim Y^\circ = 3$. Assume that the family f° is of maximal variation, i.e., that $\mathrm{Var}(f^\circ) = \dim Y^\circ$. Then $\kappa(Y^\circ) \neq 0$.*

The proof of Proposition 5.1 follows the line of argumentation outlined in Section 4.1. We prove that the Picard number of a suitable minimal model cannot be one, thereby exhibiting a fibre space structure to which induction can be applied. The presentation follows [KK08c, Sect. 9].

5.2. Sketch of proof

In essence, we follow the line of argument sketched in Section 4.1. We argue by contradiction, i.e., we maintain the assumptions of Proposition 5.1 and assume in addition that $\kappa(Y^\circ) = 0$.

5.2.1. Setup of notation As before, choose a smooth compactification $Y \supseteq Y^\circ$ such that $D := Y \setminus Y^\circ$ is a divisor with only simple normal crossings. Let $\lambda : Y \dashrightarrow Y_\lambda$ be the rational map obtain by a run of the minimal model program for the pair (Y, D) and set $D_\lambda := \lambda_*(D)$. The following is then known to hold.

(1) The variety Y_λ is normal and $\mathbb{Q}$-factorial.
(2) The variety Y_λ is log terminal. The pair (Y_λ, D_λ) is dlt.
(3) There exists a number m' such that $m'\big(K_{Y_\lambda} + D_\lambda\big) \equiv 0$. In particular, the divisor $K_{Y_\lambda} + D_\lambda$ is numerically trivial.

By Viehweg-Zuo's Theorem 3.1, there exists a number $m > 0$ and a big invertible sheaf $\mathcal{A} \subseteq \operatorname{Sym}^m \Omega^1_Y(\log D)$. As we have seen in Proposition 4.7, this induces a reflexive sheaf $\mathcal{A}_\lambda \subseteq \operatorname{Sym}^{[m]} \Omega^1_{Y_\lambda}(\log D_\lambda)$ of rank one and Kodaira-Iitaka dimension $\kappa(\mathcal{A}_\lambda) = \dim Y_\lambda$.

5.2.2. The Harder-Narasimhan filtration of $\Omega^{[1]}_{Y_\lambda}(\log D_\lambda)$ As in Section 4.1 above, we employ the Harder-Narasimhan filtration to obtain additional information about the space Y_λ.

Claim 5.2. *The divisor D_λ is not empty.*

Proof. For simplicity, we prove Claim 5.2 only in case where the canonical divisor K_{Y_λ} is Cartier, and where the space Y_λ therefore has only canonical singularities. For a proof in the general setup, the same line of argumentation applies after passing to a global index-one cover. We argue by contradiction and assume that $D_\lambda = 0$.

As before, let $C \subseteq Y_\lambda$ be a general complete intersection curve in the sense of Mehta-Ramanathan, cf. [HL97, Sect. II.7]. Since the general complete intersection curve C avoids the singular locus of Y_λ, we obtain that the restricted sheaf of Kähler differentials $\Omega^1_{Y_\lambda}|_C$ as well as its dual $\mathcal{T}_{Y_\lambda}|_C$, the restriction of the tangent sheaf, are locally free. Further, the numerical triviality of $K_{Y_\lambda} \equiv K_{Y_\lambda} + D_\lambda$ implies that

$$K_{Y_\lambda}.C = c_1\big(\Omega^{[1]}_{Y_\lambda}(\log D_\lambda)\big).C = c_1\big(\operatorname{Sym}^{[m]} \Omega^1_{Y_\lambda}(\log D_\lambda)\big).C = 0.$$

On the other hand, since $\mathcal{A}_\lambda$ is big, we have that $c_1(\mathcal{A}_\lambda).C > 0$. As in the proof of Proposition 4.1, this implies that the restricted sheaves $\Omega^1_{Y_\lambda}|_C$ as well as its dual $\mathcal{T}_{Y_\lambda}|_C$, are not semistable. The maximal destabilising subsheaf of $\mathcal{T}_{Y_\lambda}|_C$ is semistable and of positive degree, hence ample. In this setup, a variant [KST07, Cor. 5] of Miyaoka's uniruledness criterion [Miy87, Cor. 8.6] applies to give the uniruledness of Y_λ. For more details on this criterion, see the survey [KS06].

To finish the argument, let $r : W \to Y_\lambda$ be a resolution of singularities. Since uniruledness is a birational property, the space W is uniruled and therefore has Kodaira-dimension $\kappa(W) = -\infty$. On the other hand, since Y_λ has only canonical singularities, the $\mathbb{Q}$-linear equivalence class of the canonical bundle K_W is given

as

$$K_W \equiv r^*(K_{Y_\lambda}) + (\text{effective, } r\text{-exceptional divisor}).$$

But because K_{Y_λ} is $\mathbb{Q}$-linearly equivalent to the trivial divisor, we obtain that $\kappa(W) \geq 0$, a contradiction. □

5.2.3. Further contractions Claim 5.2 implies that $K_{Y_\lambda} \equiv -D_\lambda$ and it follows that for any rational number $0 < \varepsilon < 1$,

$$\kappa\big(K_{Y_\lambda} + (1-\varepsilon)D_\lambda\big) = \kappa\big(\varepsilon K_{Y_\lambda}\big) = \kappa\big(Y_\lambda\big) = -\infty. \tag{5.3}$$

Now choose one ε and run the log minimal model program for the dlt pair $\big(Y_\lambda, (1-\varepsilon)D_\lambda\big)$. This way one obtains morphisms and birational maps as follows

$$Y_\lambda \underset{\text{minimal model program}}{\overset{\mu}{\dashrightarrow}} Y_\mu \xrightarrow[\text{Mori fibre space}]{\pi} Z.$$

Again, let $D_\mu := \mu_*(D_\lambda)$ be the cycle-theoretic image of D_λ. The main properties of Y_μ and D_μ are summarised as follows.

(1) The variety Y_μ is normal and $\mathbb{Q}$-factorial.
(2) The variety Y_μ is log terminal. The pair $\big(Y_\mu, (1-\varepsilon)D_\mu\big)$ is dlt.
(3) The divisor $K_{Y_\mu} + D_\mu$ is numerically trivial.
(4) There exists a reflexive sheaf $\mathcal{A}_\mu \subseteq \operatorname{Sym}^{[m]} \Omega^1_{Y_\mu}(\log D_\mu)$ of rank one and Kodaira-Iitaka dimension $\kappa(\mathcal{A}_\mu) = \dim Y_\mu$.

In fact, more is true.

Claim 5.4. *The pair (Y_μ, D_μ) is log canonical.*

Proof. Since $K_{Y_\lambda} + D_\lambda \equiv 0$, some positive multiples of K_{Y_λ} and $-D_\lambda$ are numerically equivalent. For any two rational numbers $0 < \varepsilon', \varepsilon'' < 1$, the divisors $K_{Y_\lambda} + (1-\varepsilon')D_\lambda$ and $K_{Y_\lambda} + (1-\varepsilon'')D_\lambda$ are thus again numerically equivalent up to a positive rational multiple.

The birational map μ is therefore a minimal model program for the pair $\big(Y_\lambda, (1-\varepsilon)D_\lambda\big)$, independently of the number ε chosen in its construction. It follows that $\big(Y_\mu, D_\mu\big)$ is a limit of dlt pairs and therefore log canonical. □

5.2.4. The fibre space structure of $\boldsymbol{Y_\mu}$ Another application of the "Harder-Narasimhan-trick" exhibits a fibre structure of Y_μ.

Claim 5.5. *The Picard-number $\rho(Y_\mu)$ is larger than one. In particular, the map $Y_\mu \to Z$ is a proper fibre space whose fibres are proper subvarieties of Y_μ.*

Proof. As before, let $C \subseteq Y_\mu$ be a general complete intersection curve. Again, the existence of the Viehweg-Zuo sheaf $\mathcal{A}_\mu$ implies that the sheaf of reflexive differentials $\Omega^{[1]}_{Y_\mu}(\log D_\mu)$ is not semistable, and contains a destabilising subsheaf $\mathcal{B}_\mu \subseteq \Omega^{[1]}_{Y_\mu}(\log D_\mu)$ with $c_1(\mathcal{B}_\mu).C > 0$. Since the intersection number $c_1(\mathcal{B}_\mu).C$

is positive, the rank r of the sheaf $\mathcal{B}_\mu$ must be strictly less than $\dim Y_\mu$, and its determinant is a subsheaf of the sheaf of logarithmic r-forms,

$$\det \mathcal{B}_\mu \subseteq \Omega^{[r]}_{Y_\mu}(\log D_\mu) \quad \text{with} \quad c_1(\det \mathcal{B}_\mu).C > 0 \quad \text{and} \quad r < \dim Y_\mu.$$

If $\rho(Y_\mu) = 1$, then the sheaf $\det \mathcal{B}_\mu$ would necessarily be $\mathbb{Q}$-ample, violating the Bogomolov-Sommese vanishing theorem for log canonical pairs, Corollary 4.14. This finishes the proof of Claim 5.5. □

Now, if $F \subset Y_\mu$ is a general fibre of π and $D_F := D_\mu \cap F$, then F is a normal curve or surface, and the pair (F, D_F) is log canonical and has Kodaira dimension $\kappa(K_F + D_F) = 0$. By [KM98, Prop. 4.11], the variety F is even $\mathbb{Q}$-factorial. It is then possible to argue by induction: assuming that Viehweg's conjecture holds for families over surfaces, one obtains that the restriction of the family f° to the strict transform $(\mu \circ \lambda)^{-1}_*(F)$ cannot be of maximal variation. Since the fibres dominate the variety, this contradicts the assumption that the family f° is of maximal variation, and therefore finishes the sketch of proof of Proposition 5.1.

The reader interested in more details is referred to [KK08c, Sect. 9], where a stronger statement is shown, proving that any family over a base manifold with $\kappa(Y^\circ) = 0$ is actually isotrivial.

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